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RESEARCH ARTICLE

Analytical Investigation of the Dynamic Response of a Timoshenko Thin-Walled Beam with Asymmetric Cross Section Under Deterministic Loads

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Abstract:

Inroduction:

The objective of the present paper is to analyze dynamic response of the Timoshenko thin-walled beam with coupled bending and torsional vibrations under deterministic loads. The governing differential equations were obtained by using Hamilton's principle. The Timoshenko beam theory was employed and the effects of shear deformations, Rotary inertia and warping stiffness were included in the present formulations. Dynamic features of underlined beam are obtained using free vibration analysis.

Methods:

For this purpose, the dynamic stiffness matrix method is used. Application of exact dynamic stiffness matrix method on the movement differential equations led to the issue of nonlinear eigenvalue problem that was solved by using Wittrick–Williams algorithm . Differential equations for the displacement response of asymmetric thin-walled Timoshenko beams subjected to deterministic loads are used for extracting orthogonality property of vibrational modes.

Results:

Finally the numerical results for dynamic response in a sample of mentioned beams is presented. The presented theory is relatively general and can be used for various kinds of deterministic loading in Timoshenko thin-walled beams.

Keywords: Thin-walled beams, Timoshenko beam theory, Exact dynamic stiffness matrix method, Wittrick–Williams algorithm, Deterministic dynamic loads, Asymmetric cross section.

1. INTRODUCTION

Lightness of thin-walled beams is one of the reasons for their high performance. These beams are widely used in structures and aerospace industries. If the cross section of the thin-walled beams is mono- symmetric or asymmetric, the center of mass and the center of shear will not fit in each other. This causes a relative complexity in the behavior of these beams. Because in such a case, the bending and torsion modes are coupled and ultimately make it difficult to precisely predict the dynamic characteristics of these beams. One of the most powerful tools for solving such problems is the use of the dynamic stiffness matrix method. This method has advantages over finite element method. In the finite element method, the characteristics of an element is extracted using assumed shape functions for that element, so they are not exact. In dynamic stiffness matrix method, the dynamic characteristics are obtained from the analytical solution of the governing differential equations, so we can say that they are exact. Over the last few years, many studies have been performed in the field of formulation of dynamic stiffness matrix (DSM) of beams. The dynamic stiffness matrix of a Timoshenko beam was investigated by Cheng [1] for the first time. Williams and Howson [2] considered the effect

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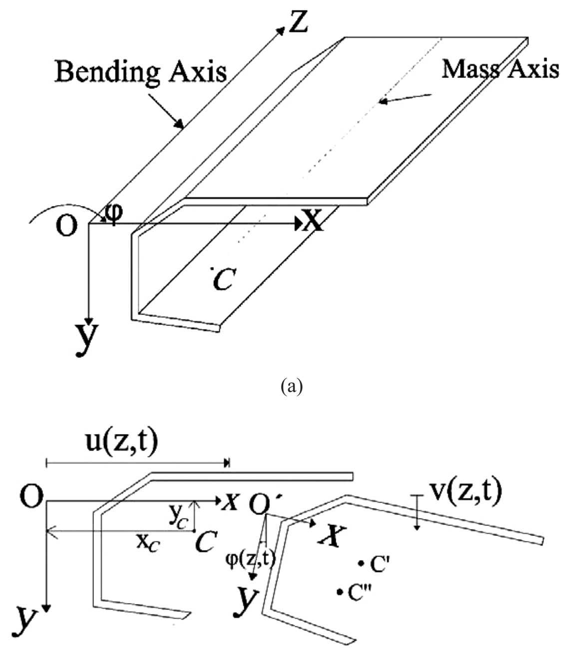
of axial load on the natural frequencies of Timoshenko beam. Banerjee [3] studied a beam with a section having one axis of symmetry and derived some explicit terms for the stiffness matrix arrays regardless of axial load effect. Banerjee and Williams [4] investigated dynamic stiffness matrix for coupled flexural-torsional vibration of Timoshenko beam. Banerjee *et al.* [5] studied warping effect on the formulation of dynamic stiffness matrix. Bercin and Tanaka [6] surveyed coupled flexural-torsional vibrations of uniform beam having single symmetric section, considering conventional support conditions. Li Jun *et al.* [7] derived the free vibrations of thin-walled Timoshenko beam under axial load, in which the effects of axial load, warping stiffness, shear deformation, and rotational inertia were taken into consideration and it was used from continuous model. Rafezy and Howson [8] derived the dynamic stiffness matrix of a three-dimensional (3D) shear-torsion beam with an asymmetric cross-section. The beam had the unusual theoretical property, so that, it allowed only for shear deformation but not bending deformation. Ghandi *et al.* [9] replaced Euler-Bernoulli theory with Timoshenko theory when the external layer of thin-walled beam is modeled and they assumed that the thin-walled part of the beam could have either open or closed section shape and would create flexural, shear, warping and Saint- Venant rigidities. Ghandi *et al.* [10] also derived the dynamic stiffness matrix of uniform beam with asymmetric cross section and elastic support under axial load. The mentioned beam consisted of an external enclosed thin-walled layer that was combined with a shear resistant filled core. Ghandi and Shiri [11] investigated the effect of the eccentricity of axial load on the natural frequencies of asymmetric thin-walled beams using exact dynamic stiffness matrix method.

Many researches have been performed in the field of the response of beams with symmetric cross-section subjected to deterministic and random dynamic loads. Eslimi-esfahani *et al*. [12] analytically investigated the dynamic response of beam with coupled flexural-torsional vibration subjected to deterministic and stochastic dynamic loads for the first time. In another study, Eslimy-Isfahany and Banerjee [13] analytically calculated dynamic response of beam with constant axial load with coupled flexural-torsional vibration under definitive and stochastic dynamic loads by using modal analysis method. Li Jun *et al*. [14] derived an explicit term for dynamic response of single symmetric Timoshenko beam subjected to stochastic excitations. Following the previous work, Li Jun *et al*. [15] derived the effects of axial load by the calculation of dynamic response of single symmetric Timoshenko beam against stochastic excitations.

In most of these researches, cross-section of the beam was mono-symmetric and Euler-Bernouli theory was used to model bending of beam. Moreover, to model beam bending, Euler-Bernouli theory is not capable of providing correct results when beams with large sections compared to their lengths or extraction of natural frequencies of higher modes are under study. In such conditions, Timoshenko beam theory in which, shear deformation and rotary inertia parameters are considered, should be employed. In this paper, considering the effect of definitive dynamic load, the analytical dynamic response of 3D flexural-torsional beam with asymmetric cross-section will be investigated by the help of exact dynamic stiffness matrix and modal analysis methods.

2. THEORY

The cross-section of the intended beam is shown in Fig. (**1**). This beam is a uniform 3D beam with asymmetric cross-section. The Timoshenko beam theory is used for modeling the bending beam. This beam has flexural rigidities of EI_x and EI_y in x - z and y - z planes, torsional warping rigidity of EI_ω, torsional Saint- Venant rigidity of G_tJ_t and shear rigidities of G_tA_{xt} and G_tA_{yt}, where G_t is the shear modulus, J_t is the section torsional constant, A_{xt} and A_{yt} are equivalent shear section in x and y directions, respectively. In Fig. (**1**), the center of gravity is denoted by C, and shear center is shown by O. The axes crossing the center of gravity and shear center are known as mass axis and bending axis, respectively. The origin of the coordinate system, is placed at O, x and y axes are in the direction of main axes of the cross-section and z axis coincides with the bending axis. The location of the point C in the co-ordinate system Oxy is given by (x_c, y_c). The beam total mass is distributed along its length as uniform distributed load and m is the beam mass per unit length. The flexural translation in the x and y directions and torsional rotation about the z-axis are represented by u (z, t), $v(z, t)$ and φ(z, t), respectively. During the translation phase $u(z, t)$ and $v(z, t)$) the shear center moves to O' and the mass center C moves to C'. During the rotation phase (φ(Z, t)), the mass center moves additionally from C' to C''. The external loads applied on the thin-walled beam include unit length forces f_x (z, t) and f_y (z, t), which are applied on the bending axis in the directions of x and y axes, respectively, unit length bending moment m_x (z, t) in the x - z plane around y axis, unit length bending moment m_x(z,t) in the x-z plane around y axis and also unit length torsion moment g (z, t) that is applied around the bending axis (Fig. **2**).



(a)

(b)

Fig. (1). (a) 3D thin-walled beam with a length of *L* and asymmetric section, (b) deformed shape of the cross-section after translational and torsional displacements.

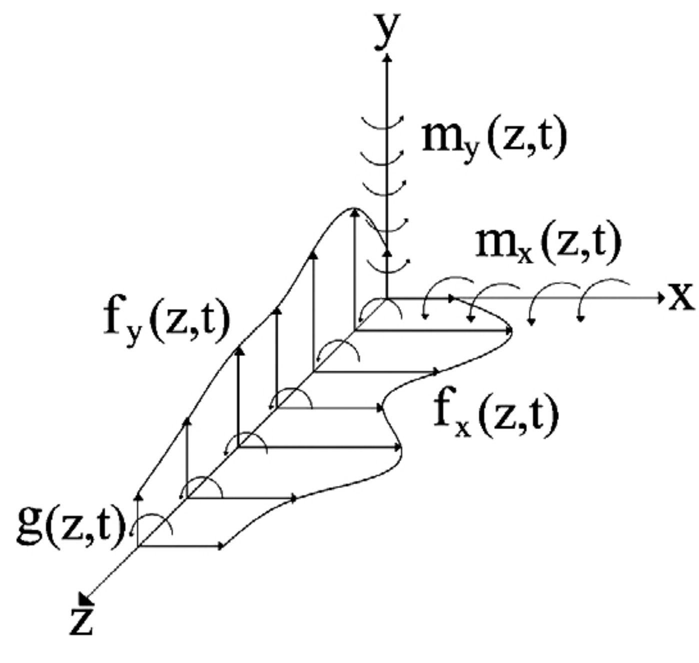


Fig. (2). Externally applied loads on a thin-walled beam.

The differential equations governing the beam moment are as five coupled partial differential equations that are defined bellow using Hamilton's principle:

$$m\frac{\partial^2 u(z,t)}{\partial t^2} - my_c\frac{\partial^2 \phi(z,t)}{\partial t^2} - G_t A_{xt}\frac{\partial^2 u(z,t)}{\partial z^2} + G_t A_{xt}\frac{\partial \theta_x(z,t)}{\partial z} = f_x(z,t) \tag{1a}$$

$$m\frac{\partial^2 v(z,t)}{\partial t^2}+mx_c\frac{\partial^2\phi(z,t)}{\partial t^2}-G_tA_{yt}\frac{\partial^2 v(z,t)}{\partial z^2}+G_tA_{yt}\frac{\partial\theta_y(z,t)}{\partial z}=f_y(z,t) \quad \textbf{(1b)}$$

$$\rho I_x\frac{\partial^2\theta_x(z,t)}{\partial t^2}-EI_x\frac{\partial^2\theta_x(z,t)}{\partial z^2}-G_tA_{xt}\frac{\partial u(z,t)}{\partial z}+G_tA_{xt}\theta_x(z,t)=m_x(z,t) \quad \textbf{(1c)}$$

$$\rho I_y\frac{\partial^2\theta_y(z,t)}{\partial t^2}-EI_y\frac{\partial^2\theta_y(z,t)}{\partial z^2}-G_tA_{yt}\frac{\partial v(z,t)}{\partial z}+G_tA_{yt}\theta_y(z,t)=m_y(z,t) \quad \textbf{(1d)}$$

$$-my_c\frac{\partial^2 u(z,t)}{\partial t^2}+mx_c\frac{\partial^2 v(z,t)}{\partial t^2}+m\,r_m^2\frac{\partial^2\phi(z,t)}{\partial t^2}-G_tJ_t\frac{\partial^2\phi(z,t)}{\partial z^2}+EI_\omega\frac{\partial^4\phi(z,t)}{\partial z^4}=g(z,t) \quad \textbf{(1e)}$$

where θ_x $(Z,\ t)$ and θ_y $(Z,\ t)$ are the rotation of the cross-section due to bending in the *x-z* and *y–z* planes, respectively. *p*, is the material density of the thin-walled beam.

Also by using Hamilton's principle, the expressions for shear forces Q_x $(z,\ t)$ and Q_y $(z,\ t)$, bending moments M_x $(z,\ t)$ and M_y $(z,\ t)$, torsional moment $T(z,\ t)$ and Bi-moment $B(z,\ t)$ can be obtained as follows:

$$M_x(z,t)=-EI_x\left(\frac{\partial\theta_x(z,t)}{\partial z}\right) \quad \textbf{(2a)}$$

$$M_y(z,t)=-EI_y\left(\frac{\partial\theta y(z,t)}{\partial z}\right) \quad \textbf{(2b)}$$

$$B(z,t)=-EI_\omega\left(\frac{\partial^2\varphi(z,t)}{\partial z^2}\right) \quad \textbf{(2c)}$$

$$Q_x(z,t)=-G_tA_{xt}\left(\frac{\partial u(z,t)}{\partial z}-\theta_x(z,t)\right) \quad \textbf{(2d)}$$

$$Q_y(z,t)=-G_tA_{yt}\left(\frac{\partial v(z,t)}{\partial z}-\theta_y(z,t)\right) \quad \textbf{(2e)}$$

$$T(z,t)=EI_w\left(\frac{\partial^3\phi(z,t)}{\partial z^3}\right)-G_tJ_t\left(\frac{\partial\phi(z,t)}{\partial z}\right) \quad \textbf{(2f)}$$

3. FREE VIBRATION ANALYSIS

In order to determine natural frequencies and vibration modes, it is required to perform the undamped free vibration analysis of the system. For this purpose, the external applying forces are considered equal to zero and thus, the above equations could be written as follows

$$m\frac{\partial^2 u(z,t)}{\partial t^2}-my_c\frac{\partial^2\phi(z,t)}{\partial t^2}-G_tA_{xt}\frac{\partial^2 u(z,t)}{\partial z^2}+G_tA_{xt}\frac{\partial\theta_x(z,t)}{\partial z}=0 \quad \textbf{(3a)}$$

$$m\frac{\partial^2 v(z,t)}{\partial t^2}+mx_c\frac{\partial^2\phi(z,t)}{\partial t^2}-G_tA_{yt}\frac{\partial^2 v(z,t)}{\partial z^2}+G_tA_{yt}\frac{\partial\theta_y(z,t)}{\partial z}=0 \quad \textbf{(3b)}$$

$$\rho I_x\frac{\partial^2\theta_x(z,t)}{\partial t^2}-EI_x\frac{\partial^2\theta_x(z,t)}{\partial z^2}-G_tA_{xt}\frac{\partial u(z,t)}{\partial z}+G_tA_{xt}\theta_x(z,t)=0 \quad \textbf{(3c)}$$

$$\rho I_y \frac{\partial^2\theta_y(z,t)}{\partial t^2} - EI_y \frac{\partial^2\theta_y(z,t)}{\partial z^2} - G_t A_{yt} \frac{\partial v(z,t)}{\partial z} + G_t A_{yt}\theta_y(z,t) = 0 \tag{3d}$$

$$-my_c \frac{\partial^2 u(z,t)}{\partial t^2} + mx_c \frac{\partial^2 v(z,t)}{\partial t^2} + m\, r_m^2 \frac{\partial^2\phi(z,t)}{\partial t^2} - G_t J_t \frac{\partial^2\phi(z,t)}{\partial z^2} + EI_\omega \frac{\partial^4\phi(z,t)}{\partial z^4} = 0 \tag{3e}$$

In order to analyze the free vibration, the answers of u (z, t), v (z, t) θ (z, t), and θ (z, t) are written as follows

$$u(z,t) = U_r(z)e^{i\omega_r t} \quad , \quad v(z,t) = V_r(z)e^{i\omega_r t} \quad , \quad \theta_x(z,t) = \Theta_{xr}(z)e^{i\omega_r t} \tag{4}$$
$$\theta_y(z,t) = \Theta_{yr}(z)e^{i\omega_r t} \quad , \quad \phi(z,t) = \Phi_r(z)e^{i\omega_r t}$$

In the above relations, r = 1,2,3... represents the vibrational mode number.

Substituting equation (4) into equation (3), gives

$$-m\,\omega_r^2 U_r(z) + my_c\omega_r^2 \Phi_r(z) - G_t A_{xt} \frac{d^2U_r(z)}{dz^2} + G_t A_{xt} \frac{d\Theta_{xr}(z)}{dz} = 0 \tag{5a}$$

$$-m\,\omega_r^2 V_r(z) - mx_c\omega_r^2 \Phi_r(z) - G_t A_{yt} \frac{d^2V_r(z)}{dz^2} + G_t A_{yt} \frac{d\Theta_{yr}(z)}{dz} = 0 \tag{5b}$$

$$-\rho I_x \omega_r^2 \Theta_{xr}(z) - EI_x \frac{d^2\Theta_{xr}(z)}{dz^2} - G_t A_{xt} \frac{dU_r(z)}{dz} + G_t A_{xt}\Theta_{xr}(z) = 0 \tag{5c}$$

$$-\rho I_y \omega_r^2 \Theta_{yr}(z) - EI_y \frac{d^2\Theta_{yr}(z)}{dz^2} - G_t A_{yt} \frac{dV_r(z)}{dz} + G_t A_{yt}\Theta_{yr}(z) = 0 \tag{5d}$$

$$my_c\omega_r^2 U_r(z) - mx_c\omega_r^2 V_r(z) - m\,r_m^2\omega_r^2 \Phi_r(z) - G_t J_t \frac{d^2\Phi_r(z)}{dz^2} + EI_\omega \frac{d^4\Phi_r(z)}{dz^4} = 0 \tag{5e}$$

By applying the dynamic stiffness matrix method on the above governing differential equations, we can obtain the natural frequencies and mode shapes. For this purpose, [9 - 11] references can be seen.

Equations (5a- d) can be rewritten as follows:

$$-m\,\omega_r^2 U_r(z) + my_c\omega_r^2 \Phi_r(z) = G_t A_{xt} \frac{d^2U_r(z)}{dz^2} - G_t A_{xt} \frac{d\Theta_{xr}(z)}{dz} \tag{6a}$$

$$-m\,\omega_r^2 V_r(z) - mx_c\omega_r^2 \Phi_r(z) = G_t A_{yt} \frac{d^2V_r(z)}{dz^2} - G_t A_{yt} \frac{d\Theta_{yr}(z)}{dz} \tag{6b}$$

$$-\rho I_x \omega_r^2 \Theta_{xr}(z) - EI_x \frac{d^2\Theta_{xr}(z)}{dz^2} = G_t A_{xt} \frac{dU_r(z)}{dz} - G_t A_{xt}\Theta_{xr}(z) \tag{6c}$$

$$-\rho I_y \omega_r^2 \Theta_{yr}(z) - EI_y \frac{d^2\Theta_{yr}(z)}{dz^2} = G_t A_{yt} \frac{dV_r(z)}{dz} - G_t A_{yt}\Theta_{yr}(z) \tag{6d}$$

4. EXTRACTION OF THE ORTHOGONALITY PROPERTIES

The most significant property of the mode shapes is that they form a set of orthogonal mathematical functions. To analyze the forced vibrations, the orthogonality condition would have to be used. The orthogonality conditions are applied to any two different modes, they cannot be applied to two modes having the same frequency. For discrete systems, the orthogonality conditions are available in all references related to the dynamics of structures. The studied beam in this paper is a distributed properties system. The vibration mode shapes derived for beams with distributed properties have orthogonality relationships equivalent to those for the discrete parameter systems. Orthogonality conditions for three-dimensional asymmetric thin-walled Timoshenko beam is derived in this paper as follows:

Substituting equation (6c) into (6a) and also equation (6d) into (6b) gives

$$-m\,\omega_r^2 U_r(z)+my_c\omega_r^2\,\Phi_r(z)=-\left[\rho I_x\omega_r^2\frac{d\Theta_{xr}(z)}{dz}+EI_x\frac{d^3\Theta_{xr}(z)}{dz^3}\right] \quad \textbf{(7a)}$$

$$-m\,\omega_r^2 V_r(z)-mx_c\omega_r^2\,\Phi_r(z)=-\left[\rho I_y\omega_r^2\frac{d\Theta_{yr}(z)}{dz}+EI_y\frac{d^3\Theta_{yr}(z)}{dz^3}\right] \quad \textbf{(7b)}$$

Multiplying (7a) by U_s (z) (sth vibrational mode) and integrating with respect to Z gives

$$-\omega_r^2\int_0^L mU_r(z)U_s(z)dz+\omega_r^2\int_0^L my_c(z)U_s(z)\phi_r(z)dz=-\int_0^L\left[\rho I_x\omega_r^2\frac{d\Theta_{xr}(z)}{dz}+EI_x\frac{d^3\Theta_{xr}(z)}{dz^3}\right]U_s(z)dz \quad \textbf{(8)}$$

If the last integration in this equation is performed by parts, it is found to give

$$-\left|\left[EI_x\frac{d^2\Theta_{xr}(z)}{dz^2}+\rho I_x\omega_r^2\Theta_{xr}(z)\right]U_s(z)\right|_0^L+\int_0^L\left[EI_x\frac{d^2\Theta_{xr}(z)}{dz^2}+\rho I_x\omega_r^2\Theta_{xr}(z)\right]\frac{dU_s(z)}{dz}dz \quad \textbf{(9)}$$

The integrated term is nil because from equations (6c) and (2d), it can be seen that the contents of the square brackets are equal to the shear force which vanishes at the boundaries z = 0 and z = L at a free end. Also at a fixed end, simply supporting the end value of U (z), it is equal to zero and then the above integrated term is nil.

$$\left|(EI_x\frac{d\Theta_{xr}(z)}{dz})\frac{dU_s(z)}{dz}\right|_0^L-\int_0^L EI_x\frac{d\Theta_{xr}(z)}{dz}\frac{d^2U_s(z)}{dz^2}dz+\omega_r^2\int_0^L\rho I_x\Theta_{xr}(z)\frac{dU_s(z)}{dz}dz \quad \textbf{(10)}$$

Following this, again the integrated term vanishes since the bracketed term is equal to bending moment which is zero at the extremities, so finally the equation (8) is expressed as follows:

$$-\omega_r^2\int_0^L mU_r(z)U_s(z)dz+\omega_r^2\int_0^L my_c\Phi_r(z)U_s(z)dz=-\int_0^L EI_x\frac{d\Theta_{xr}(z)}{dz}\frac{d^2U_s(z)}{dz^2} \quad \textbf{(11)}$$
$$+\omega_r^2\int_0^L\rho I_x\Theta_{xr}(z)\frac{dU_s(z)}{dz}dz$$

Similarly, the solving stages begin with equation (7a) rewritten for the sth mode and multiplied throughout by U_r (z) (rth vibrational mode), and by following the same steps it gives,

$$-\omega_s^2\int_0^L mU_s(z)U_r(z)dz+\omega_s^2\int_0^L my_c\Phi_s(z)U_r(z)dz=-\int_0^L EI_x\frac{d\Theta_{xs}(z)}{dz}\frac{d^2U_r(z)}{dz^2}dz \quad \textbf{(12)}$$
$$+\omega_s^2\int_0^L\rho I_x\Theta_{xs}(z)\frac{dU_r(z)}{dz}dz$$

If the last two equations are subtracted, it can now be observed that

$$(\omega_r^2-\omega_s^2)\int_0^L mU_r(z)U_s(z)dz-\omega_r^2\int_0^L my_c\Phi_r(z)U_s(z)dz+\omega_s^2\int_0^L my_c\Phi_s(z)U_r(z)dz= \tag{13}$$

$$\int_0^L EI_x\left(\frac{d\Theta_{xr}(z)}{dz}\frac{d^2U_s(z)}{dz^2}-\frac{d\Theta_{xs}(z)}{dz}\frac{d^2U_r(z)}{dz^2}\right)dz-\omega_r^2\int_0^L \rho I_x\Theta_{xr}(z)\frac{dU_s(z)}{dz}dz$$

$$+\omega_s^2\int_0^L \rho I_x\Theta_{xs}(z)\frac{dU_r(z)}{dz}dz$$

By performing the similar operation for equation (7b), the following result is finally obtained

$$(\omega_r^2-\omega_s^2)\int_0^L mV_r(z)V_s(z)dz+\omega_r^2\int_0^L mx_c\Phi_r(z)V_s(z)dz-\omega_s^2\int_0^L mx_c\Phi_s(z)V_r(z)dz= \tag{14}$$

$$\int_0^L EI_y\left(\frac{d\Theta_{yr}(z)}{dz}\frac{d^2V_s(z)}{dz^2}-\frac{d\Theta_{ys}(z)}{dz}\frac{d^2V_r(z)}{dz^2}\right)dz-\omega_r^2\int_0^L \rho I_y\Theta_{yr}(z)\frac{dV_s(z)}{dz}dz$$

$$+\omega_s^2\int_0^L \rho I_y\Theta_{ys}(z)\frac{dV_r(z)}{dz}dz$$

By dividing equation (6c) to G_tA_{xt} and (6d) to G_tA_{yt} find that

$$\frac{dU_r(z)}{dz}=\Theta_{xr}(z)-\left[\left\{EI_x\frac{d^2\Theta_{xr}(z)}{dz^2}+\rho I_x\omega_r^2\Theta_{xr}(z)\right\}\Big/ \mathrm{G_tA_{xt}}\right] \tag{15a}$$

$$\frac{dV_r(z)}{dz}=\Theta_{yr}(z)-\left[\left\{EI_y\frac{d^2\Theta_{yr}(z)}{dz^2}+\rho I_y\omega_r^2\Theta_{yr}(z)\right\}\Big/ \mathrm{G_tA_{yt}}\right] \tag{15b}$$

When equation (15a) is differentiated with respect to z, multiplied by $EI_x\dfrac{d\Theta_{xs}(z)}{dz}$ and then integrated with respect to z, finally it gives

$$\int_0^L EI_x\frac{d\Theta_{xs}(z)}{dz}\frac{d^2U_r(z)}{dz^2}dz=\int_0^L EI_x\frac{d\Theta_{xs}(z)}{dz}\frac{d\Theta_{xr}(z)}{dz}dz+\int_0^L\frac{EI_x\dfrac{d^2\Theta_{xr}(z)}{dz^2}EI_x\dfrac{d^2\Theta_{xs}(z)}{dz^2}}{G_tA_{xt}}dz \tag{16}$$

$$+\omega_r^2\int_0^L\frac{\rho I_x\,\Theta_{xr}(z)EI_x\dfrac{d^2\Theta_{xs}(z)}{dz^2}}{G_tA_{xt}}dz$$

When the suffices *r* and *s* in this last equation are interchanged, it is found that

$$\int_0^L EI_x \frac{d\Theta_{xr}(z)}{dz} \frac{d^2U_s(z)}{dz^2} dz = \int_0^L EI_x \frac{d\Theta_{xr}(z)}{dz} \frac{d\Theta_{xs}(z)}{dz} dz + \int_0^L \frac{EI_x \frac{d^2\Theta_{xs}(z)}{dz^2} EI_x \frac{d^2\Theta_{xr}(z)}{dz^2}}{G_t A_{xt}} dz + \omega_s^2 \int_0^L \frac{\rho I_x \Theta_{xs}(z) EI_x \frac{d^2\Theta_{xr}(z)}{dz^2}}{G_t A_{xt}} dz \tag{17}$$

By subtracting equation (16) from (17), the following relation is obtained.

$$\int_0^L EI_x \left(\frac{d\Theta_{xr}(z)}{dz^2} \frac{d^2U_s(z)}{dz^2} - \frac{d\Theta_{xs}(z)}{dz^2} \frac{d^2U_r(z)}{dz^2}\right) dz = \omega_s^2 \int_0^L \frac{\rho I_x \Theta_{xs}(z) EI_x \frac{d^2\Theta_{xr}(z)}{dz^2}}{G_t A_{xt}} dz - \omega_r^2 \int_0^L \frac{\rho I_x \Theta_{xr}(z) EI_x \frac{d^2\Theta_{xs}(z)}{dz^2}}{G_t A_{xt}} dz \tag{18}$$

By performing the similar operation for equation (15b), the following result is finally obtained

$$\int_0^L EI_y \left(\frac{d\Theta_{yr}(z)}{dz} \frac{d^2V_s(z)}{dz^2} - \frac{d\Theta_{ys}(z)}{dz} \frac{d^2V_r(z)}{dz^2}\right) dz = \omega_s^2 \int_0^L \frac{\rho I_y \Theta_{ys}(z) EI_y \frac{d^2\Theta_{yr}(z)}{dz^2}}{G_t A_{yt}} dz - \omega_r^2 \int_0^L \frac{\rho I_y \Theta_{yr}(z) EI_y \frac{d^2\Theta_{ys}(z)}{dz^2}}{G_t A_{yt}} dz \tag{19}$$

Multiplying equation (15a) throughout by $\omega_s^2 \rho I_x \Theta_{xs}(z)$ and integrating with respect to z, give,

$$\omega_s^2 \int_0^L \rho I_x \Theta_{xs}(z) \frac{dU_r(z)}{dz} = \omega_s^2 \int_0^L \rho I_x \Theta_{xs}(z) \Theta_{xr}(z) dz - \omega_s^2 \int_0^L \frac{EI_x \frac{d^2\Theta_{xr}(z)}{dz^2} \rho I_x \Theta_{xs}(z)}{G_t A_{xt}} dz - \omega_r^2 \omega_s^2 \int_0^L \frac{(\rho I_x)^2 \Theta_{xr}(z) \Theta_{xs}(z)}{G_t A_{xt}} dz \tag{20}$$

Interchanging *r* and *s* gives:

$$\omega_r^2 \int_0^L \rho I_x \Theta_{xr}(z) \frac{dU_s(z)}{dz} dz = \omega_r^2 \int_0^L \rho I_x \Theta_{xr}(z) \Theta_{xs}(z) dz - \omega_r^2 \int_0^L \frac{EI_x \frac{d^2\Theta_{xs}(z)}{dz^2} \rho I_x \Theta_{xr}(z)}{G_t A_{xt}} dz - \omega_s^2 \omega_r^2 \int_0^L \frac{(\rho I_x)^2 \Theta_{xs}(z) \Theta_{xr}(z)}{G_t A_{xt}} dz \tag{21}$$

When these last two equations are subtracted, they give

$$\omega_s^2\int_0^L \rho I_x\Theta_{xs}(z)\frac{dU_r(z)}{dz}-\omega_r^2\int_0^L \rho I_x\Theta_{xr}(z)\frac{dU_s(z)}{dz}dz=(\omega_s^2-\omega_r^2)\int_0^L \rho I_x\Theta_{xr}(z)\Theta_{xs}(z)dz \tag{22}$$

$$-\omega_s^2\int_0^L\frac{EI_x\dfrac{d^2\Theta_{xr}(z)}{dz^2}\rho I_x\Theta_{xs}(z)}{G_tA_{xt}}dz+\omega_r^2\int_0^L\frac{EI_x\dfrac{d^2\Theta_{xs}(z)}{dz^2}\rho I_x\Theta_{xr}(z)}{G_tA_{xt}}dz$$

Multiplying equation (15b) throughout by $\omega_s^2\rho I_y\Theta_{ys}(z)$ and integrating with respect to z, give:

$$\omega_s^2\int_0^L \rho I_y\Theta_{ys}(z)\frac{dV_r(z)}{dz}dz=\omega_s^2\int_0^L \rho I_y\Theta_{yr}(z)\Theta_{ys}(z)dz-\omega_s^2\int_0^L\frac{EI_y\dfrac{d^2\Theta_{yr}(z)}{dz^2}\rho I_y\Theta_{ys}(z)}{G_tA_{yt}}dz \tag{23}$$

$$-\omega_r^2\omega_s^2\int_0^L\frac{(\rho I_y)^2\Theta_{yr}(z)\Theta_{ys}(z)}{G_tA_{yt}}dz$$

When the suffices r and s in this last equation are interchanged, the equation becomes

$$\omega_r^2\int_0^L \rho I_y\Theta_{yr}(z)\frac{dV_s(z)}{dz}dz=\omega_r^2\int_0^L \rho I_y\Theta_{ys}(z)\Theta_{yr}(z)dz-\omega_r^2\int_0^L\frac{EI_y\dfrac{d^2\Theta_{ys}(z)}{dz^2}\rho I_y\Theta_{yr}(z)}{G_tA_{yt}}dz \tag{24}$$

$$-\omega_r^2\omega_s^2\int_0^L\frac{(\rho I_y)^2\Theta_{ys}(z)\Theta_{yr}(z)}{G_tA_{yt}}dz$$

Subtraction now reveals that,

$$\omega_s^2\int_0^L \rho I_y\Theta_{ys}(z)\frac{dV_r(z)}{dz}dz-\omega_r^2\int_0^L \rho I_y\Theta_{yr}(z)\frac{dV_s(z)}{dz}dz=(\omega_s^2-\omega_r^2)\int_0^L \rho I_y\Theta_{yr}(z)\Theta_{ys}(z)dz \tag{25}$$

$$-\omega_s^2\int_0^L\frac{EI_y\dfrac{d^2\Theta_{yr}(z)}{dz^2}\rho I_y\Theta_{ys}(z)}{G_tA_{yt}}dz+\omega_r^2\int_0^L\frac{EI_y\dfrac{d^2\Theta_{ys}(z)}{dz^2}\rho I_y\Theta_{yr}(z)}{G_tA_{yt}}dz$$

If equation (18) and (22) are now added together, they give

$$(\omega_s^2-\omega_r^2)\int_0^L \rho I_x\Theta_{xr}(z)\Theta_{xs}(z)dz=\int_0^L EI_x\left(\frac{d\Theta_{xr}(z)}{dz}\frac{d^2U_s(z)}{dz^2}-\frac{d\Theta_{xs}(z)}{dz}\frac{d^2U_r(z)}{dz^2}\right)dz \tag{26}$$

$$+\omega_s^2\int_0^L \rho I_x\Theta_{xs}(z)\frac{dU_r(z)}{dz}dz-\omega_r^2\int_0^L \rho I_x\Theta_{xr}(z)\frac{dU_s(z)}{dz}dz$$

Also if equations (19) and (25) are now added together, they give

$$(\omega_s^2-\omega_r^2)\int_0^L \rho I_y \Theta_{yr}(z)\Theta_{ys}(z)dz=\int_0^L EI_y\left(\frac{d\Theta_{yr}(z)}{dz}\frac{d^2V_s(z)}{dz^2}-\frac{d\Theta_{ys}(z)}{dz}\frac{d^2V_r(z)}{dz^2}\right)dz$$
$$+\omega_s^2\int_0^L \rho I_y\Theta_{ys}(z)\frac{dV_r(z)}{dz}dz-\omega_r^2\int_0^L \rho I_y\Theta_{yr}(z)\frac{dV_s(z)}{dz}dz \quad \textbf{(27)}$$

Comparison of equation (26) with that of equation (13) reveals that

$$(\omega_s^2-\omega_r^2)\int_0^L \rho I_x\Theta_{xr}(z)\Theta_{xs}(z)dz=(\omega_r^2-\omega_s^2)\int_0^L mU_r(z)U_s(z)dz-\omega_r^2\int_0^L my_c\Phi_r(z)U_s(z)dz$$
$$+\omega_s^2\int_0^L my_c\Phi_s(z)U_r(z)dz \quad \textbf{(28)}$$

Also the comparison of equation (27) with (14) gives

$$(\omega_s^2-\omega_r^2)\int_0^L \rho I_y\Theta_{yr}(z)\Theta_{ys}(z)dz=(\omega_r^2-\omega_s^2)\int_0^L mV_r(z)V_s(z)dz+\omega_r^2\int_0^L mx_c\Phi_r(z)V_s(z)dz$$
$$-\omega_s^2\int_0^L mx_c\Phi_s(z)V_r(z)dz \quad \textbf{(29)}$$

In this step, equation (5e) is written for the r[th] and s[th] modes; these are multiplied throughout by Φ_s (Z) and Φ_r (Z) integrated with respect to z. Finally it gives

$$(\omega_r^2-\omega_s^2)\int_0^L m\,r_m^2\,\Phi_r(z)\Phi_s(z)dz+mx_c\left[\omega_r^2\int_0^L V_r(z)\Phi_s(z)dz-\omega_s^2\int_0^L V_s(z)\Phi_r(z)dz\right]$$
$$-my_c\left[\omega_r^2\int_0^L U_r(z)\Phi_s(z)dz-\omega_s^2\int_0^L U_s(z)\Phi_r(z)dz\right]=0 \quad \textbf{(30)}$$

Equations (28), (29) and (30) may be added together to give

$$(\omega_r^2-\omega_s^2)\int_0^L\Big[\big(m\,r_m^2\Phi_r(z)\Phi_s(z)+mV_r(z)V_s(z)+mU_r(z)U_s(z)+\rho I_x\Theta_{xr}(z)\Theta_{xs}(z)$$
$$+\rho I_y\Theta_{yr}(z)\Theta_{ys}(z)\big)+mx_c\big(V_s(z)\Phi_r(z)+V_r(z)\Phi_s(z)\big)$$
$$-my_c\big(U_s(z)\Phi_r(z)+U_r(z)\Phi_s(z)\big)\Big]dz=0 \quad \textbf{(31)}$$

In the above equation, if r = s, then the following equation is obtained.

$$(\omega_r^2-\omega_r^2)\int_0^L\Big\{m(r_m^2\left[\Phi_r(z)\right]^2+\left[V_r(z)\right]^2+\left[U_r(z)\right]^2)+\rho I_x\left[\Theta_{xr}(z)\right]^2+\rho I_y\left[\Theta_{yr}(z)\right]^2$$
$$+2mx_cV_r(z)\Phi_r(z)-2my_cU_r(z)\Phi_r(z)\Big\}dz=0 \quad \textbf{(32)}$$

The above relation is satisfied because, $(\omega_r^2-\omega_r^2)=0$, so the result of the integral is a constant value as follows,

$$\int_0^L \Big\{ m\left(r_m^2\left[\Phi_r(z)\right]^2 + \left[V_r(z)\right]^2 + \left[U_r(z)\right]^2\right) + \rho I_x\left[\Theta_{xr}(z)\right]^2 + \rho I_y\left[\Theta_{yr}(z)\right]^2 \tag{33}$$

$$+2mx_c V_r(z)\Phi_r(z) - 2my_c U_r(z)\Phi_r(z)\Big\}dz = \mu_r$$

For $r \neq s$, so that $(\omega_r^2\text{-}\omega_r^2) \neq 0$ therefore, to derive equation (31), we must have the following relation:

$$\int_0^L \big[m\, r_m^2 \Phi_r\Phi_s + mV_rV_s + mU_rU_s + \rho I_x\Theta_{xr}\Theta_{xs} + \rho I_y\Theta_{yr}\Theta_{ys} + mx_c(V_s\Phi_r + V_r\Phi_s) \tag{34}$$

$$-my_c(U_s\Phi_r + U_r\Phi_s)\big]dz = 0$$

Equations (33) and (34) can be shown as the following general form

$$(\omega_r^2 - \omega_s^2)\int_0^L \big[(m\, r_m^2\Phi_r\Phi_s + mV_rV_s + mU_rU_s) + \rho I_x\Theta_{xr}\Theta_{xs} + \rho I_y\Theta_{yr}\Theta_{ys} + mx_c(V_s\Phi_r + V_r\Phi_s) \tag{35}$$

$$-my_c(U_s\Phi_r + U_r\Phi_s)\big]dz = \mu_r\delta_{sr}$$

The above equation shows the orthogonality condition for different mode shapes of the thin-walled Timoshenko beams with asymmetric cross-section. μ is the generalized mass in the r[th] mode, and δ_{sr} the Kronecker delta function defined as follows:

$$\delta_{sr} = \begin{cases} 1 & s = r \\ 0 & s \neq r \end{cases} \tag{36}$$

With the free vibration natural frequencies, mode shapes and orthogonality condition described above, we can calculate dynamic response of the Timoshenko thin-walled beam under deterministic loads.

5. DYNAMIC RESPONSE ANALYTICAL CALCULATION

Now, the partial differential equations (1) are taken into consideration that are required to be solved for the applied external forces of f_x (z, t), f_y (z, t), m_x (z, t), m_y (z, t), and g (z, t). Assuming that the eigenvalue problem is solved for extracting natural frequencies and modes, the response against the applied loads is obtained from the linear combination of the modes as follows:

$$u(z,t) = \sum_{r=1}^{\infty} q_r(t)\, U_r(z) \tag{37a}$$

$$v(z,t) = \sum_{r=1}^{\infty} q_r(t)\, V_r(z) \tag{37b}$$

$$\theta_x(z,t) = \sum_{r=1}^{\infty} q_r(t)\, \Theta_{xr}(z) \tag{37c}$$

$$\theta_y(z,t) = \sum_{r=1}^{\infty} q_r(t)\, \Theta_{yr}(z) \tag{37d}$$

$$\varphi(z,t) = \sum_{r=1}^{\infty} q_r(t)\, \Phi_r(z) \tag{37e}$$

In the above equations, q_r (t) is the modal coordinate (time coordinate) of the rth mode. As a result, the responses u (z, t), v (z, t), θ_x (z, t), θ_y (z, t) and ϕ_x (z, t) are defined as the total participation effects of each mode. The r^{th} term in the series of equation (37) represents the participation rate of the r^{th} mode.

Substituting equations (37) into equations (1) and the introduction of the non-dimensional variable $\xi = \frac{z}{L}$ then yields

$$\sum_{r=1}^{\infty}\left[mU_r\ddot{q}_r(t) - my_c\Phi_r\ddot{q}_r(t) - G_tA_{xt}\frac{d^2U_r}{dz^2}q_r(t) + G_tA_{xt}\frac{d\Theta_{xr}}{dz}q_r(t)\right] = f_x(\xi,t) \tag{38a}$$

$$\sum_{r=1}^{\infty}\left[mV_r\ddot{q}_r(t) + mx_c\Phi_r\ddot{q}_r(t) - G_tA_{yt}\frac{d^2V_r}{dz^2}q_r(t) + G_tA_{yt}\frac{d\Theta_{yr}}{dz}q_r(t)\right] = f_y(\xi,t) \tag{38b}$$

$$\sum_{r=1}^{\infty}\left[\rho I_x\Theta_{xr}\ddot{q}_r(t) - EI_x\frac{d^2\Theta_{xr}}{dz^2}q_r(t) - G_tA_{xt}\frac{dU_r}{dz}q_r(t) + G_tA_{xt}\Theta_{xr}q_r(t)\right] = m_x(\xi,t) \tag{38c}$$

$$\sum_{r=1}^{\infty}\left[-my_cU_r\ddot{q}_r(t) + mx_cV_n\ddot{q}_r(t) + mr_m^2\Phi_r\ddot{q}_r(t) - G_tJ_t\frac{d^2\Phi_r}{dz^2}q_r(t) + EI_w\frac{d^4\Phi_r}{dz^4}q_r(t)\right] = g(\xi,t) \tag{38d}$$

$$\sum_{r=1}^{\infty}\left[\rho I_y\Theta_{yr}\ddot{q}_r(t) - EI_y\frac{d^2\Theta_{yr}}{dz^2}q_r(t) - G_tA_{yt}\frac{dV_r}{dz}q_r(t) + G_tA_{yt}\Theta_{yr}q_r(t)\right] = m_y(\xi,t) \tag{38e}$$

Substituting equations (5) into equations (38) gives:

$$\sum_{r=1}^{\infty}\left[m(U_r - y_c\Phi_r)\ddot{q}_r(t) + m\omega_r^2(U_r - y_c\Phi_r)q_r(t)\right] = f_x(\xi,t) \tag{39a}$$

$$\sum_{r=1}^{\infty}\left[m(V_r + x_c\Phi_r)\ddot{q}_r(t) + m\omega_r^2(V_r + x_c\Phi_r)q_r(t)\right] = f_y(\xi,t) \tag{39b}$$

$$\sum_{r=1}^{\infty}\left[\rho I_x\Theta_{xr}\ddot{q}_r(t) + \rho I_x\omega_r^2\Theta_{xr}q_r(t)\right] = m_x(\xi,t) \tag{39c}$$

$$\sum_{r=1}^{\infty}\left[\rho I_y\Theta_{yr}\ddot{q}_r(t) + \rho I_y\omega_r^2\Theta_{yr}q_r(t)\right] = m_y(\xi,t) \tag{39d}$$

$$\sum_{r=1}^{\infty}\left[m\omega_r^2(x_cV_r - y_cU_r + r_m^2\Phi_r)q_r(t) + m(x_cV_r - y_cU_r + r_m^2\Phi_r)\ddot{q}_r(t)\right] = g(\xi,t) \tag{39e}$$

In this step, each sentence of (39a) in U_s , each sentence of (39b) in V_s, each sentence of (39c) in Θ_{xs}, each sentence of (39d) in Θ_{ys} and each sentence of (39e) are multiplied by Φ_s and integrated throughout the beam. After adding the obtained equations, the equation becomes

$$\sum_{r=1}^{\infty} q_r(t)\int_0^L m\omega_r^2\left[U_rU_s+V_rV_s+r_m^2\Phi_r\Phi_s-y_c(\Phi_rU_s+U_r\Phi_s)+x_c(\Phi_rV_s+V_r\Phi_s)\right]dz \tag{40}$$

$$+\sum_{r=1}^{\infty} \ddot{q}_r(t)\int_0^L m\left[U_rU_s+V_rV_s+r_m^2\Phi_r\Phi_s-y_c(\Phi_rU_s+U_r\Phi_s)+x_c(\Phi_rV_s+V_r\Phi_s)\right]dz$$

$$+\sum_{r=1}^{\infty} q_r(t)\int_0^L \omega_r^2(\rho I_x\Theta_{xr}\Theta_{xs}+\rho I_y\Theta_{yr}\Theta_{ys})dz+\sum_{r=1}^{\infty}\ddot{q}_r(t)\int_0^L(\rho I_x\Theta_{xr}\Theta_{xs}+\rho I_y\Theta_{yr}\Theta_{ys})dz$$

$$=\int_0^L\left[f_x(\xi,t)U_s+f_y(\xi,t)V_s+m_x(\xi,t)\Theta_{xs}+m_y(\xi,t)\Theta_{ys}+g(\xi,t)\Phi_s\right]dz$$

The above equation is as follows after simplifying and changing the order of integral and sign $\sum$

$$\sum_{r=1}^{\infty} q_r(t)\int_0^L m\omega_r^2\left[U_rU_s+V_rV_s+r_m^2\Phi_r\Phi_s-y_c(\Phi_rU_s+U_r\Phi_s)+x_c(\Phi_rV_s+V_r\Phi_s)\right]dz \tag{41}$$

$$+\sum_{r=1}^{\infty} \ddot{q}_r(t)\int_0^L m\left[U_rU_s+V_rV_s+r_m^2\Phi_r\Phi_s-y_c(\Phi_rU_s+U_r\Phi_s)+x_c(\Phi_rV_s+V_r\Phi_s)\right]dz$$

$$+\sum_{r=1}^{\infty} q_r(t)\int_0^L \omega_r^2(\rho I_x\Theta_{xr}\Theta_{xs}+\rho I_y\Theta_{yr}\Theta_{ys})dz+\sum_{r=1}^{\infty}\ddot{q}_r(t)\int_0^L(\rho I_x\Theta_{xr}\Theta_{xs}+\rho I_y\Theta_{yr}\Theta_{ys})dz$$

$$=\int_0^L\left[f_x(\xi,t)U_s+f_y(\xi,t)V_s+m_x(\xi,t)\Theta_{xs}+m_y(\xi,t)\Theta_{ys}+g(\xi,t)\Phi_s\right]dz$$

Regarding the property of the orthogonality of modes given in (35), it was observed that for all values of $r \neq s$, the value of the two integrals on the left of the above equation is eliminated, only for $r = s$ the following equation will remain:

$$q_r(t)\int_0^L m\omega_r^2\left[[U_r]^2+[V_r]^2+r_m^2[\Phi_r]^2-2y_cU_r\Phi_r+2x_cV_r\Phi_r\right]dz \tag{42}$$

$$+\ddot{q}_r(t)\int_0^L m\left[[U_r]^2+[V_r]^2+r_m^2[\Phi_r]^2-2y_c\Phi_rU_r+2x_cV_r\Phi_r\right]dz$$

$$+q_r(t)\int_0^L \omega_r^2\left[\rho I_x[\Theta_{xr}]^2+\rho I_y[\Theta_{yr}]^2\right]dz+\ddot{q}_r(t)\int_0^L\left[\rho I_x[\Theta_{xr}]^2+\rho I_y[\Theta_{yr}]^2\right]dz$$

$$=\int_0^L\left[f_x(z,t)U_r+f_y(z,t)V_r+m_x(z,t)\Theta_{xr}+m_y(z,t)\Theta_{yr}+g(z,t)\Phi_r\right]dz$$

The above equation can be expressed in the following form

$$(\ddot{q}_r(t)+\omega_r^2 q_r(t))=\frac{1}{\mu_r}\int_0^L \left[f_x(z,t)U_r+f_y(z,t)V_r+m_x(z,t)\Theta_{xr}+m_y(z,t)\Theta_{yr}+g(z,t)\Phi_r\right]dz \tag{43}$$

By introducing the following parameters,

$$\frac{1}{\mu_r}\int_0^L f_x(z,t)U_r(z)dz=F_{xr}(t) \tag{44a}$$

$$\frac{1}{\mu_r}\int_0^L f_y(z,t)V_r(z)dz=F_{yr}(t) \tag{44b}$$

$$\frac{1}{\mu_r}\int_0^L m_x(z,t)\Theta_{xr}(z)dz=M_{xr}(t) \tag{44c}$$

$$\frac{1}{\mu_r}\int_0^L m_y(z,t)\Theta_{yr}(z)dz=M_{yr}(t) \tag{44d}$$

$$\frac{1}{\mu_r}\int_0^L g(z,t)\Phi_r dz=G_r(t) \tag{44e}$$

Now, equation (43) is rewritten as follows

$$\ddot{q}_r(t)+\omega_r^2 q_r(t)=F_{xr}(t)+F_{yr}(t)+M_{xr}(t)+M_{yr}(t)+G_r(t) \tag{45}$$

Therefore, there are an infinite number of equations similar to equation (45) and one equation for each mode. The partial differential equation (1) for unknown functions *u* (z, t), *v* (z, t) θ_x (z, t) θ_y *(z, t)* and Φ (z, t) are transferred into an infinite set of ordinary differential equations (45) in terms of $q_r(t)$ unknowns.

For the applied dynamic loads of f_x *(z, t)*, f_y *(z, t)*, m_x *(z, t)*, m_y *(z, t)* and *g(z, t)* the unknown system functions *u* (z, t), *v* (z, t), θ_x *(z, t)*, θ_y *(z, t)* and ϕ (z, t) could be determined by solving the modal equations in terms of q_r (t). The equation of each mode is independent to the other modes; thus, it could be solved separately. For solving equation (45), which is similar to the form of movement equation for single degree of freedom (SDOF) system, Duhamel's integral is used. Therefore, the answer of equation (45) is defined as:

$$q_r(t)=(A_r\cos\omega_r t+B_r\sin\omega_r t)+\frac{1}{\omega_r}\int_0^t (G_r(\tau)+M_{yr}(\tau)+M_{xr}(\tau)+F_{yr}(\tau)+F_{xr}(\tau))\sin\omega_r(t-\tau)d\tau \tag{46}$$

After determining and q_r (t), by using equations (37) and (46), system response to arbitrary dynamic forces f_x *(z, t)*, f_y *(z, t)*, m_x *(z, t)*, m_y *(z, t)* and *g (z, t)* is as follows

$$u(z,t)=\sum_{r=1}^{\infty}U_r(z)\Big((A_r\cos\omega_r t+B_n\sin\omega_r t)+\frac{1}{\omega_r}\int_0^t (G_r(\tau)+M_{yr}(\tau)+M_{xr}(\tau)+F_{yr}(\tau)+F_{xr}(\tau)) \times\sin\omega_r(t-\tau)d\tau\Big) \tag{47a}$$

$$v(z,t)=\sum_{r=1}^{\infty}V_r(z)\Big((A_r\cos\omega_r t+B_r\sin\omega_r t)+\frac{1}{\omega_r}\int_0^t (G_r(\tau)+M_{yr}(\tau)+M_{xr}(\tau)+F_{yr}(\tau)+F_{xr}(\tau)) \times\sin\omega_r(t-\tau)d\tau\Big) \tag{47b}$$

$$\theta_x(z,t)=\sum_{r=1}^{\infty}\Theta_{xr}(z)\Big((A_r\cos\omega_r t+B_r\sin\omega_r t)+\frac{1}{\omega_r}\int_0^t(G_r(\tau)+M_{yr}(\tau)+M_{xr}(\tau)+F_{yr}(\tau)+F_{xr}(\tau))$$
$$\times\sin\omega_r(t-\tau)d\tau\Big) \qquad \textbf{(47c)}$$

$$\theta_y(z,t)=\sum_{r=1}^{\infty}\Theta_{yr}(z)\Big((A_r\cos\omega_r t+B_r\sin\omega_r t)+\frac{1}{\omega_r}\int_0^t(G_r(\tau)+M_{yr}(\tau)+M_{xr}(\tau)+F_{yr}(\tau)+F_{xr}(\tau))$$
$$\times\sin\omega_r(t-\tau)d\tau\Big) \qquad \textbf{(47d)}$$

$$\phi(z,t)=\sum_{r=1}^{\infty}\Phi_r(z)\Big((A_r\cos\omega_r t+B_r\sin\omega_r t)+\frac{1}{\omega_r}\int_0^t(G_r(\tau)+M_{yr}(\tau)+M_{xr}(\tau)+F_{yr}(\tau)+F_{xr}(\tau))$$
$$\times\sin\omega_r(t-\tau)d\tau\Big) \qquad \textbf{(47e)}$$

The obtained responses can be used for any arbitrary deterministic loading. In the following, the response against deterministic harmonic load is calculated using equations (47) for instance.

In this part, it was assumed that the centralized harmonic forces having F_{xi} amplitudes are applied in the direction of x axis in the points Z_i = a_i the forces with F_{yi} amplitudes are applied in the direction of y axis in the points Z_i = b_i, bending moments having M_{xi} amplitudes are applied in x-z plane around y axis and in the points Z_i = C_i, bending moments with M_{yi} amplitudes are applied in y-z plane around x axis and in the points $Z_i = d_i$, and torsional moments with G_i amplitudes are applied around z axis and in the points Z_i = e_i, where i = 1,2,3,.....,N. The mentioned applied loads are defined as follows:

$$f_{xi}(z,t)=F_{xi}\,\delta(z-a_i)\sin\omega_i t \qquad \textbf{(48a)}$$

$$f_{yi}(z,t)=F_{yi}\,\delta(z-b_i)\sin\omega_i t \qquad \textbf{(48b)}$$

$$m_{xi}(z,t)=M_{xi}\,\delta(z-c_i)\sin\omega_i t \qquad \textbf{(48c)}$$

$$m_{yi}(z,t)=M_{yi}\,\delta(z-d_i)\sin\omega_i t \qquad \textbf{(48d)}$$

$$g_i(z,t)=G_i\,\delta(z-e_i)\sin\omega_i t \qquad \textbf{(48e)}$$

where, ω_i is the rotational frequency of the applied loads.

In this state, by using equations (44), the generalized loads functions become,

$$F_{xir}(z,t)=\frac{1}{\mu_r}F_{xi}\,U_r(a_i)\sin\omega_i t \qquad \textbf{(49a)}$$

$$F_{yir}(z,t)=\frac{1}{\mu_r}F_{yi}\,V_r(b_i)\sin\omega_i t \qquad \textbf{(49b)}$$

$$M_{xir}(z,t)=\frac{1}{\mu_r}M_{xi}\,\Theta_{xr}(c_i)\sin\omega_i t \qquad \textbf{(49c)}$$

$$M_{yir}(z,t)=\frac{1}{\mu_r}M_{yi}\,\Theta_{yr}(d_i)\sin\omega_i t \qquad \textbf{(49d)}$$

$$G_{ir}(z,t)=\frac{1}{\mu_r}G_i\,\Phi_r(\mathrm{e}_i)\sin\omega_i t \qquad \textbf{(49e)}$$

Finally, the response of $u\ (z, t)$, $v\ (z, t)$, $\theta_x\ (z, t)$, $\theta_y\ (z, t)$ and $\phi(z, t)$ to assumed applying loads is as follows

$$u(z,t)=\sum_{r=1}^{\infty}U_r(z)\,(A_r\cos\omega_r t+B_r\sin\omega_r t+\sum_{r=1}^{\infty}U_r(z)\sum_{i=1}^{N}\frac{1}{\mu_r(\omega_r^2-\omega_i^2)}(F_{xi}U_r(a_i)+F_{yi}V_r(b_i)+ \tag{50a}$$
$$M_{xi}\Theta_{xr}(c_i)+M_{yi}\Theta_{yr}(d_i)+G_i\Phi_r(e_i))\sin\omega_i t$$

$$v(z,t)=\sum_{r=1}^{\infty}V_r(z)\,(A_r\cos\omega_r t+B_r\sin\omega_r t+\sum_{r=1}^{\infty}V_r(z)\sum_{i=1}^{N}\frac{1}{\mu_r(\omega_r^2-\omega_i^2)}(F_{xi}U_r(a_i)+F_{yi}V_r(b_i)+ \tag{50b}$$
$$M_{xi}\Theta_{xr}(c_i)+M_{yi}\Theta_{yr}(d_i)+G_i\Phi_r(e_i))\sin\omega_i t$$

$$\theta_x(z,t)=\sum_{r=1}^{\infty}\Theta_{xr}(z)\,(A_r\cos\omega_r t+B_r\sin\omega_r t+\sum_{r=1}^{\infty}\Theta_{xr}(z)\sum_{i=1}^{N}\frac{1}{\mu_r(\omega_r^2-\omega_i^2)}(F_{xi}U_r(a_i)+F_{yi}V_r(b_i)+ \tag{50c}$$
$$M_{xi}\Theta_{xr}(c_i)+M_{yi}\Theta_{yr}(d_i)+G_i\Phi_r(e_i))\sin\omega_i t$$

$$\theta_y(z,t)=\sum_{r=1}^{\infty}\Theta_{yr}(z)\,(A_r\cos\omega_r t+B_r\sin\omega_r t+\sum_{r=1}^{\infty}\Theta_{yr}(z)\sum_{i=1}^{N}\frac{1}{\mu_r(\omega_r^2-\omega_i^2)}(F_{xi}U_r(a_i)+F_{yi}V_r(b_i)+ \tag{50d}$$
$$M_{xi}\Theta_{xr}(c_i)+M_{yi}\Theta_{yr}(d_i)+G_i\Phi_r(e_i))\sin\omega_i t$$

$$\phi(z,t)=\sum_{r=1}^{\infty}\Phi_r(z)\,(A_r\cos\omega_r t+B_r\sin\omega_r t+\sum_{r=1}^{\infty}\Phi_r(z)\sum_{i=1}^{N}\frac{1}{\mu_r(\omega_r^2-\omega_i^2)}(F_{xi}U_r(a_i)+F_{yi}V_r(b_i)+ \tag{50e}$$
$$M_{xi}\Theta_{xr}(c_i)+M_{yi}\Theta_{yr}(d_i)+G_i\Phi_r(e_i))\sin\omega_i t$$

6. NUMERICAL RESULTS

The following example is presented in order to validate the formulation proposed in the present paper.

In this example, a thin-walled cantilever beam having semicircular section with one axis of symmetry is investigated. The physical and geometrical specifications of the studied section are as follows (Fig. **3**):

$$E=68.9\times10^9\,Pa \quad , \quad I_y=9.26\times10^{-8}m^4 \quad , \quad I_x=1.77\times10^{-8}m^4$$
$$I_\omega=1.52\times10^{-12}m^6 \quad , \quad r_m^2=5.998\times10^{-4}m^2 \quad , \quad m=.0835kg.m^{-1}$$
$$L=0.82m \quad , \quad y_c=0.0m \quad , \quad x_c=0.0155m \quad , \quad G=26.5\times10^9\,Pa \quad , \quad J=1.64\times10^{-9}m^4$$

In this example, it was assumed that unit harmonic forces with unit amplitude are applied to the tip of the cantilever beam and the transitional bending displacement, rotational and torsional angles were calculated at the cantilever beam tip under the applied harmonic load with different frequencies. For calculating the response, the first five modes of vibration were used. Therefore, the first five bending–torsion coupled natural frequencies and vibrating modes were first calculated with the help of the dynamic stiffness matrix. The calculated natural frequencies are presented in Table **1**.

Table 1. The first five natural frequencies.

f (*HZ*)	ω(*Rad/Sec*)	Frequency order
1	399.0379	63.5089
2	863.4747	137.4263
3	1733.26	275.8580
4	3023.24	481.1625
5	4020.25	639.8936

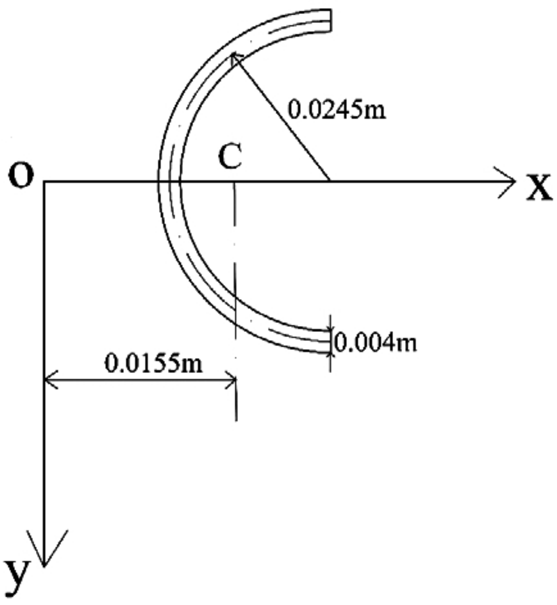


Fig. (3). Beam cross-section used in numerical example.

Then, for each mode, the values of μ were calcluted. For the first five modes, the μ as follows:

$$\mu_1 = 4.5944, \quad \mu_2 = 4.5453, \quad \mu_3 = 2.5601, \quad \mu_4 = 3.6127, \quad \mu_5 = 4.7357$$

By using equations (50) and considering $F_x = 0$, $M_x = 0$, $G = 0$ and $F_y = 1$ as well as the location of the point, b = 0.82m, the intended responses were calculated at the cantilever edge point and plotted in the semi-logarithmic diagrams of Figs. (**4**, **5**, and **6**). In the mentioned figures, the absolute magnitudes of the obtained values were considered on the vertical axis, which is a logarithmic axis.

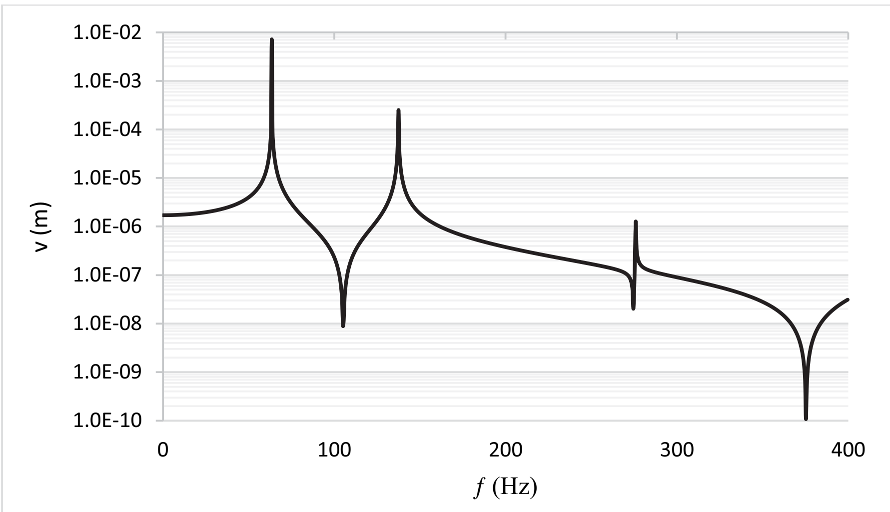


Fig. (4). Dynamic transitional bending displacement of thin-walled beam at its tip for different frequencies of the applied load.

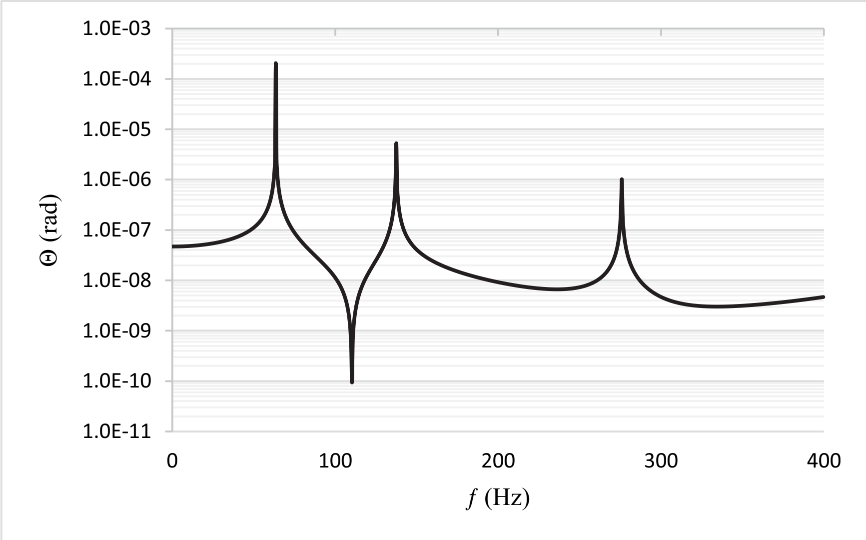


Fig. (5). Dynamic rotational bending displacement of thin-walled beam at its tip for different frequencies of the applied load.

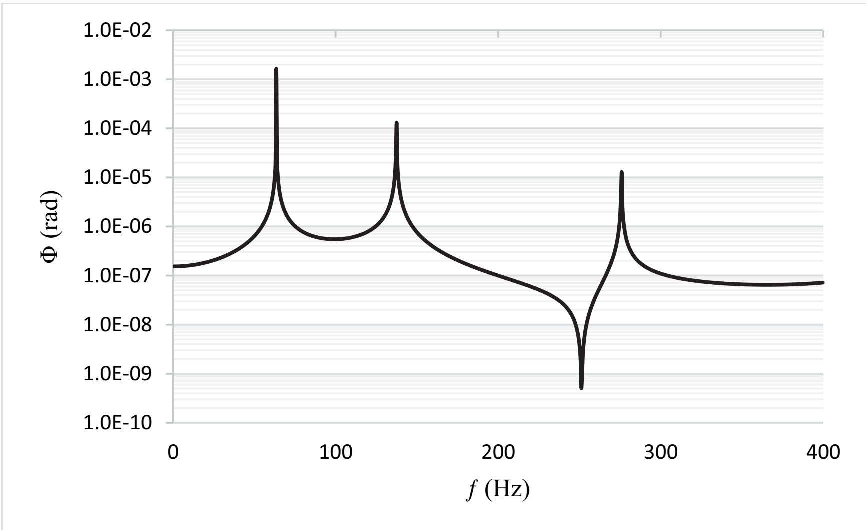


Fig. 6. Dynamic torsional displacement of thin-walled beam at its tip for different frequencies of the applied load.

CONCLUSION

In the present study, an analytical method is proposed for determining the dynamic response of asymmetric thin-walled beams subjected to different types of centralized and distributed definitive dynamic loads. In order to solve vibration problems, the study used accurate dynamic stiffness matrix. Since the dynamic stiffness matrix is derived from analytical solution of the differential equations of movement, it makes it possible to calculate natural frequencies and vibrating modes accurately and without any loss in the precision. The natural frequencies and mode shapes were obtained by using Wittrick–Williams algorithm.

Due to the general shape of the beam (*i.e.* a perfect asymmetric section), the mass and shear centers were not coincident and thus, flexural and torsional vibrations were observed to be dependent on each other. Accordingly, determining the analytical response of the 3D thin-walled beam with asymmetric section against dynamic deterministic loads is considered as a very complicated problem. Using the introduced dynamic stiffness matrix in combination with modal analysis method, this complexity can be addressed.

By using the formulation presented in this paper, the dynamic response of members with arbitrary asymmetric section can be derived also with different boundary conditions under any arbitrary applied definitive dynamic loading at various points.

CONSENT FOR PUBLICATION

Not applicable.

CONFLICT OF INTEREST

The authors declare no conflict of interest, financial or otherwise.

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