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# Remarks on the Current Theory of Shear Strength of Variable Depth Beams

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**Abstract:** Though still in use today, the method of the effective shearing force to evaluate the maximum shear stress in variable depth beams does not stand close scrutiny. It can lead to overestimating the shearing strength of these beams, although it is suggested as a viable procedure by many otherwise excellent codes of practice worldwide. This paper should help to put the record straight. It should warn the practitioner against the general inadequacy of such a method and prompt the drafters of structural concrete codes of practice into acting to eliminate this erroneous though persisting designing procedure.

**Key Words:** Variable depth beams, Effective shearing force, Tapered beams, Building codes.

## INTRODUCTION

The three simply supported beams shown in Fig. (**1**) all have the same span and carry the same load. Their shape is different, though. Beam (a) is a constant depth beam, while beams (b) and (c) are just two different instances of a variable depth beam. All three beams share the same bending moment and shearing force diagrams (also shown in the picture). This is so, because in the present case these diagrams are determined from equilibrium conditions only.

At any cross section of any beam, bending moment M and shearing force V must always be related to each other by the well known equation

$$\frac{dM}{dz} = V \qquad \dots \qquad (1)$$

where z denotes the coordinate along the beam axis. This means that the shearing force and bending moment diagrams cannot be assigned independently of each other. Equation (1) is an equilibrium condition. As such it applies irrespectively of whether the beam is elastic, plastic, composite, fissured etc; be it made of steel, or reinforced concrete or any other material. To assume any shearing force V, different than the one given by eq. (1), would therefore be tantamount as violating the equilibrium of the beam. Yet, when it comes to evaluating the bearing capacity in shear of these three beams, the current procedure has it that we should refer to three different shearing force diagrams which depend on how the depth of the beam varies with z, in spite of the fact that the bending moment diagram, and hence the derivative dM/dz, is the same for all the three beams.

The standard argument to support this practice runs as follows. At any cross-section of a variable depth beam the flexural stresses in compression and in tension are respectively assumed to result in a sloped force if they are relevant to a sloped side of the beam. Accordingly, the component of such a force which is normal to the beam axis is supposed to increase or decrease the internal shear needed to equilibrate the shearing force acting at the considered cross section. This is what is stated, almost literally, in Sect. R 11.1.1.2 of the American Concrete Institute Code [1] and represents the accepted wisdom concerning this topic.

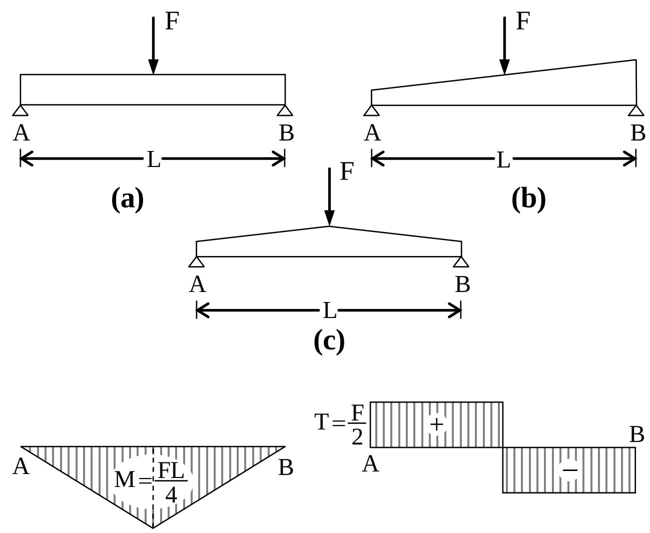


**Fig. (1).** Irrespectively of their shape, the three simply supported beams (**a**), (**b**) and (**c**) share the same bending moment and shearing force diagrams (shown beneath them).

In fact it will be proved presently that the above arguments are not correct. Despite this, they have remained unchallenged for well over seven decades till today, leaving their mark on generations of building codes all over the world, and finding their way into an incredible sequence of otherwise excellent textbooks, cf. e.g. Park and Pauley [2] or MacGregor [3] to quote just a pair of instances for the many available. The aim of the present paper is to put the record straight. We shall track the origin of such a fallacy and show how was it that it could firmly establish itself within the primary literature on the argument.

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Of course the topic is not merely academic. If the slope of the intrados at the built-in ends of a beam is steep enough, the current procedure can easily lead to the conclusion that no special shear reinforcement is needed. In fact, it is not difficult to envisage realistic cases where a more precise finite element analysis dictates that under the same conditions the maximum shear stress by far exceeds the one calculated by the standard method.

## ORIGIN OF THE PROCEDURE AND FACT OF THE MATTER

Apparently, all began with the classical textbook by Bleich [4] dating back to 1932. In Ch.16 of that book, Bleich extended the well known Jourawski's method, still in use today, to determine the mean value of the shear stress acting on any chord of any cross section of variable depth beams. Bleich's analysis also included the effect of a normal force acting on the beam cross sections. It also covered the general case in which both the intrados and the extrados of the beam were sloped. For simplicity's sake, we shall refer to the particular case in which the normal force vanishes and either the intrados or the extrados –not both– is sloped. These restrictions are not crucial to the validity of the arguments that follow, but make for formulae that are simpler and readily comparable with those in current use in the engineering literature on the argument.

In his paper Bleich uses the notation "max $\tau_m$" to indicate the value of the shear stress $\tau$ at the centroid C of the beam cross sections. The index *m* appended to $\tau$ stands for *mittelwert* or average value. This is consistent with the fact that, generally speaking, the value of $\tau$ that is obtained from Jourawski's method is in fact the average value over the chord through C, parallel to the neutral axis. Through a very accurate analysis, Bleich ended up with a formula (cf. eq. (20), Ch. 16, of Bleich [4]) giving the average shear stress on that chord. Bleich's formula has appeared over and over again ever since in the engineering literature on the argument. Quite correctly, it accounts for the shear stress that, in a variable depth beam, is also produced by the bending moment M and by the axial force N, in addition to that produced by V. The case in which the axial force can be neglected does often occur in the engineering practice. In this case Bleich's formula can be –and most frequently is– rewritten in the simple form:

$$\max \tau_m = \frac{V^*}{b_n D} \quad \ldots \quad (2)$$

(cf. e.g. Park and Pauley [2], eqs (7.6)-(7.6a), for N=0). The notation here is the following. The quantity D is the so-called internal lever arm, given by

$$D = \frac{J_x}{S_{max}} \quad \ldots \quad (3)$$

where $J_x$ is the inertia moment of the beam cross section with respect to the neutral axis, while $S_{max}$ is the first moment of area of the section above the neutral axis, calculated with respect to the neutral axis itself. The quantity V* appearing in eq. (2) is usually referred to as the *equivalent* or *effective* shearing force and is defined as

$$V^* = V - \frac{1}{a} M \quad \ldots \quad (4)$$

where 1/a is given by:

$$\frac{1}{a} = \frac{1}{D}\frac{dD}{dz} \quad \ldots \quad (5)$$

Finally, in eq. (2) the quantity $b_n$ denotes the length of the chord through C, parallel to the neutral axis.

Fig. (**2**) shows a segment of a beam and its cross section. It provides an illustration of all the quantities related to this problem. For simplicity's sake, a beam of rectangular cross section is considered. The gist of the arguments to be presented applies, however, to cross sections of any shape.

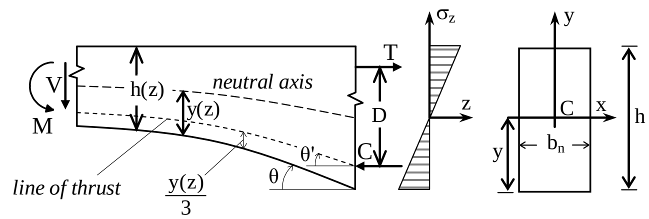


**Fig. (2).** A segment of a variable depth beam, illustrating the main symbols adopted.

From eqs (3)-(5) and (1) it can be verified that the amplitude of the effective shearing force V* turns out to be smaller than that of the true shearing force V whenever the depth of the beam increases in the same direction in which the amplitude of the bending moment increases. Otherwise, V* is greater than V.

The case of constant slope occurs frequently in the applications. In this case we have that

$$\frac{dD}{dz} = \mathrm{tg}\,\theta' \quad \ldots \quad (6)$$

where θ' is the slope of the so called *line of thrust*, joining the points of application of the resultant of the normal stresses on the part of the beam cross section that lies between the neutral axis and the sloped side of the beam, cf. Fig. (**2**). Of course for small enough slopes we have that $\theta' \cong \theta$, the angle θ being the angle that the sloped side of the beam forms with the level one.

So far so good, but for one small detail. It concerns the prefix "max" preceding $\tau_m$ in eq. (2). It is all right for constant depth beams (provided of course that their thickness is constant, or nearly so, about the centroid of the cross section and that, moreover, the latter is not too weird). For variable depth beams, however, $\tau_m$ does not reach its maximum value at C (not even in the case in which the beam cross sections is rectangular!). In other words, although eq. (2) does indeed provide the right value of the shear stress at the centroid of the cross section, that value is not the maximum shear stress of the considered cross section if the depth of the beam is variable. As a consequence, the prefix "max" attached to $\tau_m$ in eq. (2) is not right if referred to such beams.

To prove this claim we simply have to give a look at the rigorous solution of the shear stress problem in variable depth beams. This solution has been known for a long time and is available in several textbooks on elasticity theory. The reader is referred to Oden [5] for a clear exposition of this topic. Further references include the classical textbooks by Timoshenko [6] (Ch. 3), by Timoshenko and Goodier [7] (Sects 35 and 41) and by L'Hermite [8] (Ch. 3, Sect. 16). Of course the rigorous solution depends on a number of factors, ranging from the shape of the beam, to the shape of its cross section and the load distribution pattern. This makes the resulting formulae rather cumbrous. Simpler formulae are obtained when the rate of change of the beam depth is not too rapid. In this case, the bending stress at each cross section is to a good approximation given by the standard flexure formula $\sigma_z = M\, y/J_x$ . Even in this case, though, a general expression of the shear stress distribution on the beam cross sections is hardly practical. Reference to particular instances is enough, however, to show how strongly the shear stress distribution in the cross sections of a variable depth beam can differ from the familiar one of constant depth beams.

Consider for instance the case of the cantilever beam of a rectangular cross section of variable depth, shown in Fig. (**3**). The intrados of the beam has a non vanishing constant slope, so that the depth of the beam varies linearly with the distance z from its free end. Denoting by h=h(z) this depth, we shall have, therefore,

$$h(z) = h_0 + \frac{h_L - h_0}{L} z = h_0 \left(1 + \frac{\alpha}{L} z\right) \qquad \dots \qquad (7)$$

Here $h_0$, $h_L$ and L have the meaning illustrated in Fig. (**3**), while

$$\alpha = \frac{h_L - h_0}{h_0} \qquad \dots \qquad (8)$$

In the case considered in Fig. (**3**), we have $h_L = 2h_0$, which makes $\alpha = 1$. The shearing force is constant along the beam axis and given by V=P=const. The correct shear stress at each cross section of the beam can accordingly be expressed as:

$$\tau_{yz} = \frac{6\,V\,\zeta}{b\,h_0^{\,2}\left[\,h(z)/h_0\,\right]^4} \left\{ 1 - \frac{\left[h(z) - h_0\right]^2}{h_0^{\,2}} - \frac{\zeta}{h_0}\left[1 - 2\,\frac{h(z) - h_0}{h_0}\right] \right\} \qquad \dots \qquad (9)$$

which rewrites in the present notation a formula by Oden, cf. [5], Sect. 5.4, eq. (d). Here, the quantity $\zeta$ is defined by:

$$\zeta = \zeta(z,y) = h(z)/2 + y \qquad \dots \qquad (10)$$

It should be noted that, as in any boundary-value problem of the elasticity theory, the solution of the stress field depends on the boundary conditions. Equation (9) refers to the case in which the shear stress applied to the cross section z=0 is distributed according to the standard theory of shear stress in rectangular cross section beams of constant depth, see formula (11) below. This does not introduce any serious restriction since, if the slope of the intrados of the beam is not too steep, the stress distribution at the end of the beam has little influence on the stress distribution at the beam cross sections which are sufficiently far from that end.

The diagrams in Fig. (**3**) show how $\tau_{zy}$ varies along the beam depth of the cross sections at z=L/2 and z=L, respectively. In these diagrams, the correct values of $\tau_{zy}$ as obtained from eq. (9) are shown in continuous lines. Dashed lines represent the value of the shear stress as calculated from the standard formula:

$$\tau_{yz} = \bar{\tau}_{yz}(y) = \frac{V\,S_x}{b\,J_x} \qquad \dots \qquad (11)$$

which applies to constant depth beams. The quantity $S_x$ appearing here is the first moment with respect to the neutral axis of that part of the area of the cross section which lies

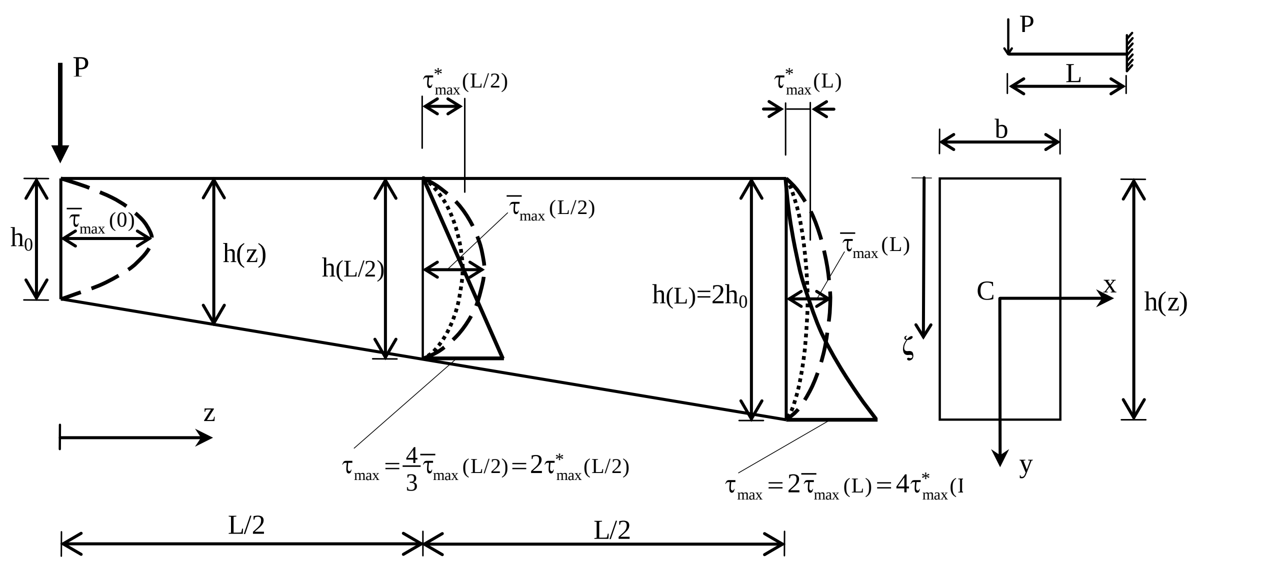


**Fig. (3).** Shear stress diagrams at selected cross sections of a beam of a variable depth. The correct diagrams are shown in solid lines. The dashed lines refer to the classical stress distribution of a constant depth beam acted upon by the true shearing force V. The dotted lines represent similar diagrams calculated for the effective shearing force V*, according to the standard method.

either above or below the chord y=const. As well-known, eq. (11) predicts that in a constant depth beam of rectangular cross section the maximum shear stress, say $\overline{\tau}_{max}$, will be reached at its centroid. Therefore, if eq. (11) is applied to find the maximum shear stress of a variable depth beam with rectangular cross sections, the result would clearly depend on z and be given by

$$\overline{\tau}_{max} = \overline{\tau}_{max}(z) = \frac{3}{2}\frac{V}{b\,h(z)} \qquad \ldots \qquad (12)$$

As Fig. (**3**) shows, however, the true maximum value of the shear stress in the beam cross sections is generally different from $\overline{\tau}_{max}$ *and* is not reached at their centroid. The same conclusion holds true, of course, if the effective shearing force V* as defined by eq. (4) is substituted for V in eq. (12). This proves that the prefix "max" appearing in eq. (2) is not appropriate.

Let us consider the matter in a little more detail. Let $\tau_c$ be the correct value of $\tau_{zy}$ at the centroid C of a cross section of the beam. This value is readily calculated from eq. (9) upon setting $\zeta$ = h(z)/2 into it. Simple algebra then shows that $\tau_c$ does indeed coincide with the value, say $\tau^*$, that is obtained eq. (11) (i.e. by setting y=0 in that equation), provided that in the same equation we substitute for V the value of V* as given by eq. (4). In particular, for rectangular cross sections this means that:

$$\tau_c = \tau^* = \frac{3}{2}\frac{V^*}{b\,h(z)} \qquad \ldots \qquad (13)$$

Such a singular coincidence of $\tau^*$ with $\tau_c$, together with the regrettable oversight of assuming that also in a variable depth beam the shear stress reached the maximum value at the centroid of the cross sections, was the source of the mistake. It led to the erroneous conclusion that in a variable depth beam the bending stress at the sloped sides of the beam resulted in a shearing component that added vectorially to the shearing force due to the external loads.

In order to better appreciate the extent of the error that can be made by using the method of the effective shearing force V* to evaluate the maximum shear stress in a variable depth beam, let us refer again to the results shown in Fig. (**3**). In that case we have that D = 2 $h_0$(1+z/L)/3, dD/dz = 2 $h_0$/(3L), a=L+z, V=P and M=P z. From eq. (4) we then get V*= 0.66 P at the cross section z=L/2. Similarly, we obtain V*=0.5 P at the built-in end of the beam (z=L). The dotted curves in Fig. (**3**) represent the shear stress that can be calculated from eq. (12) once the value of V* is substituted for V. Of course, these curves reach their maximum value at the centroid of the cross section. It is seen that for z=L/2 and z=L these maxima underestimate by a factor 2 and 4, respectively, the true maximum shear calculated from formula (9) at the same cross sections. The method of the effective shearing force to evaluate the maximum shear stress in a variable depth beam is, therefore, untenable. It is certainly true that it provides the correct value of the shear stress at the centroid of the beam cross sections. However, that value is of little use because it can be quite different from the maximum shear stress at the same sections.

## FROM A LUCKY THEORETICAL COINCIDENCE TO AN UNTENABLE PRACTICE

The dubious notion of effective shearing force we have been discussing so far refers to homogeneous elastic beams. How is it then that the same notion ended up so firmly rooted in current practice when it comes to calculating the ultimate (i.e. post-elastic) shear strength of reinforced concrete beams of variable depth? Apparently, the answer lies in a combination of a lucky coincidence, an unfortunate mistake and a fallacious interpretation resulting from the two. We already spotted the lucky coincidence: The value of $\tau_{zy}$ provided by the right-hand side of eq. (2) turns out to exactly coincide with the value of the shear stress at the centroid of the beam cross sections, as rigorously obtained from elasticity theory. This is so, at least, whenever the shape of the cross sections is such that the shear stress at the neutral axis is constant, which is the case for most of the cross sections of practical interest, including the rectangular ones. (For more general sections, the above coincidence applies to the average values of $\tau_{zy}$ along the neutral axis).

The mistake was to assume that the shear stress at the centroid of a cross section of a variable depth beam reached a maximum. As already pointed out, this can only be valid for beams of constant depth. For variable depth beams, the theory of elasticity shows that the maximum shear stress is not attained at the centroid, but rather at a point that is different for different cross sections and can even reach the sloped side of the beam (cf. Fig. **3**). The "culprit" here is that word "max" preceding $\tau_m$ in Bleich's formula (2).

The fallacious interpretation entered upon the stage when a physical explanation for the wrong formula (2) was sought. Misled by the deceptive simplicity of that formula, the elementary notion of action and reaction was forgotten. It was admitted that the tensile/compressive bending stress acting on the part of the beam cross section lying between the sloped side of the beam and the neutral axis resulted in a net non-vanishing component normal to the beam axis. The quantity -M/a appearing in eq. (4) was taken as the right value of this component. The fact that in doing so one could obtain the correct value of the shear stress at the centroid of the cross section was regarded as a proof of that conjecture. Thus, the effective shearing force V* appearing in eq. (2) was seen as resulting from the composition of V and the vertical component $\Delta$V of a hypothetical sloped force resulting from the bending stresses at the sloped side(s) of the beam. Being relevant to the stress resultant, such a composition was independent of the details concerning the particular stress distribution along the considered cross section. This meant that it was also independent of the constitutive equations of the material making up the beam. Thus, the way to extending that interpretation beyond the elastic range into the realm of limit analysis was paved.

So cogent and intuitive the notion of effective shearing force must have appeared to be, that it was forgotten that the coincidence of the shear stress given by eq. (2) and the correct value of the shear stress at the centroid of the cross section only applied to homogeneous elastic material. The concept of effective shearing force was thus arbitrarily extended beyond the elastic range. The consequence is that we now have a physically unsound, potentially harmful approach to the shear strength of reinforced concrete beams of variable depth, embodied in almost every building code of practice worldwide (cf., e.g., Sect. 11.1.1.2 of the already quoted American Concrete Institute [1] for the USA and Ch. 6.2 of Eurocode 2 [9] for all the countries belonging to the European Union).

## SOME FURTHER EVIDENCE AGAINST THE CURRENT APPROACH

Lest tradition should prevail over reason, we shall now present four paradoxes resulting from the current approach to shear strength in variable depth beams.

### Paradox 1

By referring to the variable depth cantilever shown in Fig. **4**(**a**), let $P_u$ denote its ultimate load. We assume that, if this load is exceeded, failure in shear will take place at the built-in section of the beam. Let's consider then the constant depth beam shown in Fig. **4**(**b**), the depth of which equals the maximum depth of the previous beam. From eqs (2)-(5) it can be almost immediately inferred that the ultimate load $\overline{P}_u$ needed to produce shear failure in beam (b) is less than $P_u$. This beam, however, is obtained from that of Fig. **4**(**a**) by adding the structural material that is shown as a dashed area in the figure, thus transforming beam (a) into a constant depth beam. The ultimate load of the beam thus transformed cannot be less than that of the variable depth beam from which it originated. This is both commonsense and also a well known consequence of a corollary of the Lower Bound Theorem of limit analysis, stating that: "*The collapse load in a structure cannot be decreased by increasing the strength of any part*" (cf., e.g., Horne [10], Sect. 1.6). However, should we evaluate the shear strength of the beam by using the effective shearing force given by eq. (4), we would be lead to the wrong conclusion that the variable depth beam in Fig. **4**(**a**) is capable of sustaining a greater shearing force than the constant depth beam in Fig. **4**(**b**). The reason for this would be that no reduction in the true shearing force can be applied to the latter beam, since factor 1/a vanishes in constant depth beams. Such a conclusion would clearly be untenable, however, in view of the above quoted corollary.

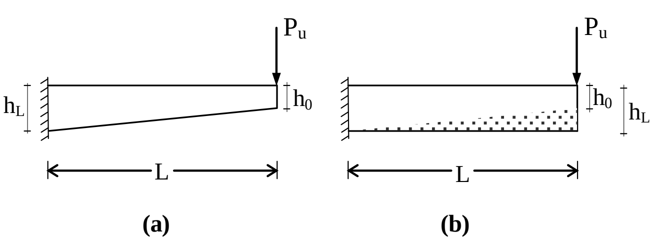


**Fig. (4).** Adding structural material (shown shadowed) to beam (a) cannot reduce its strength.

### Paradox 2

Take the simply supported variable depth beam of Fig. (**5**). Its shearing force diagram is shown underneath the beam. The shearing force $V_S$ at midspan of the beam is $V_S = R_A$ and remains constant across the midspan section. However, the effective shearing force V*, as calculated from eq. (2), would be greater or lesser than $V_S$, depending on whether it is calculated by referring to the left or to the right half of the beam. Clearly, since no force is applied to the midspan section, these values of the effective shearing force would not be consistent with the equilibrium of the beam.

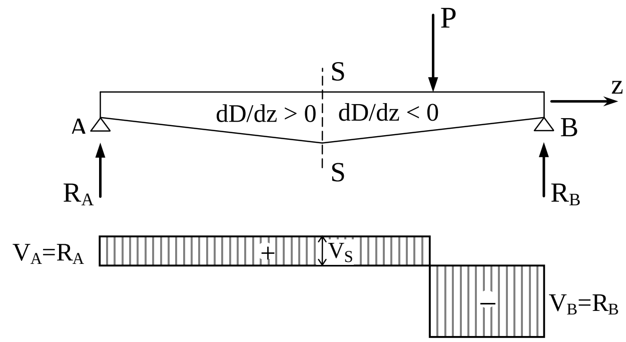


**Fig. (5).** The true shearing force V is continuous across section S-S, whereas the effective shearing force V* is not, though no force is applied at that section.

### Paradox 3

Consider a generic cross section, say S-S, of the variable depth beam of Fig. (**6**). According to the traditional interpretation of eq. (2), the vertical component of the sloped compression force C′ resulting from bending and applied to this section from the beam on its left, would reduce the shearing force acting at the same section by the amount $\Delta V = C \tan \theta'$. But, what about the vertical component of the opposite compression force -C′ applied to the same section from the right-hand side part of the beam? Being downwardly directed, this vertical component equilibrates the upward component $\Delta V$ mentioned above. No net reduction of the shearing force V acting on the section can therefore result from a sloped intrados, if the latter is rectilinear. (Clearly, this argument does not apply to the case in which the beam intrados is curved or at the points of the intrados where the slope is discontinuous. But then any effect of curvature or slope discontinuity falls outside of the standard approach. The latter only involves the angle that the sides of the beam form with the beam axis and is, therefore, independent of how this angle varies along the beam).

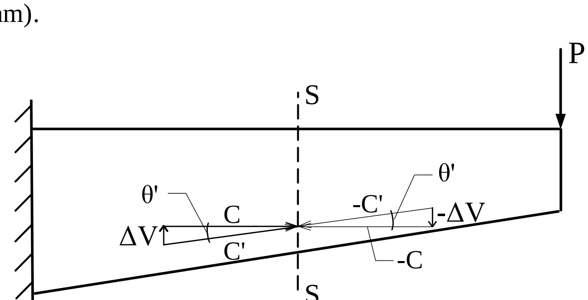


**Fig. (6).** The "sloped" compression force C′ acting from the left of section S-S is opposite to the one (-C′) that is acting from the right. Thus, no net vertical component can result from them.

### Paradox 4

This case is the dual of that leading to Paradox 1 discussed above. Consider the constant depth beam of Fig. **7**(**a**) and assume that, under the triangular load shown there, the ultimate shear strength is reached at its built-in end. Should eq. (2) be valid, we could improve the load bearing capacity of the beam by reshaping it as indicated in Fig. **7**(**b**). In this way, we could increase the ultimate strength of the beam by decreasing the strength of some of its parts. This is clearly unrealistic and, above all, in contrast with the corollary of the Lower Bound Theorem of limit analysis, already quoted when discussing Paradox 1.

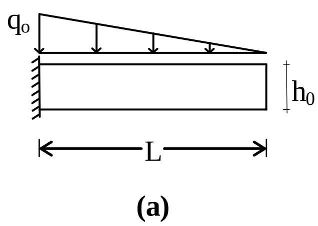


**Fig. (7).** Removing structural material from beam (**a**) cannot lead to a beam (**b**) of a greater strength.

## CONCLUSION

The standard procedure to calculate the maximum shear stress in a variable depth beam replaces the true shearing force with an effective shearing force which is different than the former. For elastic, homogeneous beams such a procedure gives the correct value of the shear stress at the centroid of the beam cross sections. The point is, however, that the shear stress is not maximum at the centroid of the sections, if the beam is of a variable depth. This is a very well established result of elasticity theory and is at variance with what applies to constant depth beams. Elasticity theory shows, moreover, that the maximum shear stress at a cross section of a variable depth beam can exceed by far the shear stress that is reached at its centroid. This makes the method of the effective shearing force incapable of determining the maximum shear stress of such a beam, even in the homogeneous elastic case.

Currently, the effective shearing force method is also used beyond the elastic range in order to calculate the shear strength of variable depth beams. This is even less admissible. Outside the elastic range, the value of the shear stress that is obtained by referring to the effective –rather than the true– shearing force does not even coincide with the correct value of the shear stress at the centroid cross sections. This means that the notion of effective shearing force is of no value whatsoever when applied to the post-elastic behavior of a beam.

Failure to recognize these points has lead to a potentially dangerous, yet widespread practice in the design of variable depth beams, which can strongly overestimate their strength in shear. Yet this practice is so firmly established that is recommended by the best building codes worldwide.

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