Optimization of Grillage-like Continuum by Triangle Plate Element

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Abstract: The volume of grillages with stress constraints is minimized. An optimal beams system or plate with reinforced ribs is obtained to present the optimal structure. A grillage-like continuum material model is adapted. Structure is analyzed by finite element method with triangle plate elements. The geometric matrix of triangle plate element in explicit formulation about area coordinates is presented. The stiffness matrix of grillage-like continuum material model is derived. The material distribution field in design domain is optimized by fully-stressed criterion. The densities and orientations of the beam or reinforced ribs at nodes in grillages are taken as design variables. The densities and orientations vary in design domain continuously. The optimal distribution fields of bend moments, flexure displacement and material are obtained simultaneously. Subsequently the discrete structures are founded based on the optimal material distribution fields. The performances of different elements are compared. The optimization procedure is accomplished by computer program automatically.

Keywords: Least-weight grillage, Topology optimization, Finite element method, Fully-stressed criterion, Triangle plate element.

1. INTRODUCTION

The grillage is kind of particular plane structure, which bear loads in landscape orientation of structure plane. It is constructed by an infinite number of rectangular cross sectional beams with infinitesimal spacing. Beam widths vary and their depths are given. The least-weight grillages, which are anisotropic continua, are named as “grillage-like continua” frequently. The optimization of least-weight grillage is a fundamental topology optimization problem. Many analytical solutions were derived by different methods [1-11]. The further reviews on the least-weight grillages were given by Rozvany and Prager, [12] Rozvany *et al.*, [13]. As we known, it is difficult to get the exact analytical solution on grillages. It is therefore meaningful to study the method to derive optimal topologies numerically.

Usually people, by various numerical topology optimization methods, intend to obtain discrete solutions like perforated plates, rather than a continuous material distribution field. However the grillages are generally continuum structures. To obtain continuous material distribution field of truss-like continuum structures, Zhou and Li [14] developed a novel numerical method basing on finite element method. This method is further generalized to the topology optimization analysis of grillages by rectangle elements [15]. In this paper, triangle elements are used to adapt irregular design domain. We derived the elastic matrix and stiffness matrix of triangle elements of grillages-like continuum material model.

In fact if there is thin plate in middle plan of grillage, the optimal grillage turn to become the optimal solid plates reinforced rib. Furthermore, the optimal grillage with stress constraint is also optimal structure with a compliance constraint or a natural frequency constraint.

2. GRILLAGE-LIKE CONTINUUM MATERIAL MODEL

It is assumed that the grillages-like continuum is isotropic and its Poisson’s ratio is zero. Therefore the relationship between stress and strain under the local coordinate system can be expressed as,

$$\bar{\boldsymbol{\sigma}} = E \cdot \mathrm{diag}[\begin{matrix} 1 & 1 & 1/2 \end{matrix}]\bar{\boldsymbol{\varepsilon}}\,, \tag{1}$$

where E is Young’s modulus. $\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{\varepsilon}}$ are the stress and strain vectors,

$$\bar{\boldsymbol{\sigma}} = [\begin{matrix} \sigma_{\bar{x}} & \sigma_{\bar{y}} & \tau_{\bar{x}\bar{y}} \end{matrix}]^{\mathrm{T}}, \tag{2}$$

$$\bar{\boldsymbol{\varepsilon}} = [\begin{matrix} \varepsilon_{\bar{x}} & \varepsilon_{\bar{y}} & \gamma_{\bar{x}\bar{y}} \end{matrix}]^{\mathrm{T}} = [\begin{matrix} \dfrac{\partial \bar{u}}{\partial \bar{x}} & \dfrac{\partial \bar{v}}{\partial \bar{y}} & \dfrac{\partial \bar{v}}{\partial \bar{x}} + \dfrac{\partial \bar{u}}{\partial \bar{y}} \end{matrix}]^{\mathrm{T}}. \tag{3}$$

The strain vector $\bar{\varepsilon}$ is related to the curvature vector $\bar{\boldsymbol{\kappa}}$ of mid-plane as,

$$\bar{\boldsymbol{\varepsilon}} = -z\bar{\boldsymbol{\kappa}}\,. \tag{4}$$

$$\bar{\boldsymbol{\kappa}} = [\begin{matrix} \dfrac{\partial^2 w}{\partial \bar{x}^2} & \dfrac{\partial^2 w}{\partial \bar{y}^2} & \dfrac{\partial^2 w}{\partial \bar{x}\partial \bar{y}} \end{matrix}]^{\mathrm{T}}, \tag{5}$$

In this paper t_1, t_2 are named as densities and they combining orientation α of beams at nodes are taken as optimum design variables.

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In the final optimum grillage-like continua, all beams are aligned along the directions of principal moments and thus no torsion deformation and related torsion moment exists in beams. Consequently the width corresponding to torsion has no effect on the final results. However it should be noted that without torsion stiffness the balance state is unstable which will cause the stiffness matrix become singular. To resolve this issue the densities corresponding to torsion is assumed as $(t_1+t_2)/2$. At the first iteration $t_1=t_2$ is assumed, which guarantees that optimization begins from an isotropic structure. With the aids of (1) and (4), the moments in the region of $\mathrm{d}\bar{y}=1$ and $\mathrm{d}\bar{x}=1$ can be calculated by integrating the stresses on the cross section,

$$\begin{aligned}\bar{\boldsymbol{M}} &= [\ M_{\bar{x}} \quad M_{\bar{y}} \quad M_{\overline{xy}}\ ]^{\mathrm{T}} = -\int_{-h/2}^{h/2} z\bar{\boldsymbol{\sigma}}\,\mathrm{diag}[\ t_1 \quad t_2 \quad (t_1+t_2)/2\ ]\,\mathrm{d}z \\ &= -\frac{1}{12}h^3\boldsymbol{D}(t_1,t_2,0)\bar{\boldsymbol{\kappa}}\end{aligned}, \quad (6)$$

where $\boldsymbol{D}(t_1,t_2,0)$ is stiffness matrix on material principal axes,

$$\boldsymbol{D}(t_1,t_2,0) = E\cdot\mathrm{diag}[\ t_1 \quad t_2 \quad (t_1+t_2)/4\ ]. \quad (7)$$

Global of Moment-Curvature Relationship

The curvatures of mid-plane and the moments on the global coordinate axes are denoted as,

$$\boldsymbol{\kappa} = [\ \frac{\partial^2 w}{\partial x^2} \quad \frac{\partial^2 w}{\partial y^2} \quad \frac{\partial^2 w}{\partial x\partial y}\ ]^{\mathrm{T}},\ \boldsymbol{M} = [\ M_x \quad M_y \quad M_{xy}\ ]^{\mathrm{T}}, \quad (8)$$

respectively. Through coordinate transformation the local terms and their global counterparts are related as,

$$\bar{\boldsymbol{\kappa}} = \boldsymbol{T}_\varepsilon\boldsymbol{\kappa},\ \bar{\varepsilon} = \boldsymbol{T}_\varepsilon\varepsilon,\ \bar{\boldsymbol{M}} = \boldsymbol{T}_\sigma\boldsymbol{M},\ \bar{\sigma} = \boldsymbol{T}_\sigma\sigma, \quad (9)$$

with

$$\boldsymbol{T}_\varepsilon(\alpha) = \boldsymbol{T}_\sigma^{-\mathrm{T}}(\alpha) = \begin{bmatrix} c^2 & s^2 & cs \\ s^2 & c^2 & -cs \\ -2cs & 2cs & c^2-s^2 \end{bmatrix}, \quad \begin{matrix} c=\cos\alpha \\ s=\sin\alpha \end{matrix} \quad (10)$$

where $\boldsymbol{T}_\varepsilon$, $\boldsymbol{T}_\sigma$ is the frame rotation matrix for strain and stress, respectively,

Combining (6), (9) and (10) yields,

$$\boldsymbol{M} = -\frac{1}{12}h^3\boldsymbol{T}_\varepsilon^{\mathrm{T}}(\alpha)\boldsymbol{D}(t_1,t_2,0)\boldsymbol{T}_\varepsilon(\alpha)\kappa = -\frac{1}{12}h^3\boldsymbol{D}(t_1,t_2,\alpha)\kappa, \quad (11)$$

where $\boldsymbol{D}(t_1,t_2,\alpha)$ is the global stiffness matrix calculated by,

$$\begin{aligned}\boldsymbol{D}(t_1,t_2,\alpha) &= \boldsymbol{T}_\varepsilon^{\mathrm{T}}(\alpha)\boldsymbol{D}(t_1,t_2,0)\boldsymbol{T}_\varepsilon(\alpha) \\ &= E\sum_{b=1}^{2}t_b\sum_{r=1}^{3}s_{br}g_r(\alpha)\boldsymbol{A}_r\end{aligned}, \quad (12)$$

$$\boldsymbol{g}(\alpha) = \begin{bmatrix} 1 & \cos 2\alpha & \sin 2\alpha \end{bmatrix},\ \boldsymbol{s} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}, \quad (13)$$

$$\boldsymbol{A}_1 = \mathrm{diag}[\ 1 \quad 1 \quad 1/2\ ],\ \boldsymbol{A}_2 = \mathrm{diag}[\ 1 \quad -1 \quad 0\ ],$$
$$\boldsymbol{A}_3 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \quad (14)$$

3. FINITE ELEMENT METHOD

Displacement Pattern

Three nodes triangle elements are used in this paper. The displacement is expressed as,

$$w = \sum_{i=1}^{3}(N_i w_i + N_{xi}\theta_{xi} + N_{yi}\theta_{yi}) \quad (15)$$

where w_i, θ_{xi} and θ_{yi} are nodal displacement and rotations, respectively; their coefficients shape function have following formula,

$$\begin{aligned} N_1 &= L_1(1+L_2L_1+L_3L_1-L_2^2-L_3^2) \\ N_{x1} &= L_1^2(b_2L_3-b_3L_2)+(b_2-b_3)L_1L_2L_3/2 \\ N_{y1} &= L_1^2(c_2L_3-c_3L_2)+(c_2-c_3)L_1L_2L_3/2 \end{aligned} \quad (16)$$

where L_i is area coordinates of any point in element and x_i, y_i the coordinates of nodes.

$$\begin{aligned} &L_i = (a_i+b_ix+c_iy)/2A,\ L_1+L_2+L_3=1 \\ &a_i = x_jy_m-x_my_j,\ b_i=y_j-y_m,\ c_i=x_m-x_j,\ i,j,m=1,2,3 \end{aligned} \quad (17)$$

Strains

Strains can be calculate by geometric matrix $\boldsymbol{B}$ and nodal displacements vector,

$$\varepsilon = \boldsymbol{B}\boldsymbol{U}_e, \quad (18)$$

Shape function (17) is the cubed function of L_i and not all of them are zeros, therefore geometric matrix $\boldsymbol{B}$ is linear function of L_i and can be written as,

$$\begin{aligned}\boldsymbol{B} &= \boldsymbol{B}(L_1,L_2,L_3,z) = z\bar{\boldsymbol{T}}(L_1\bar{\boldsymbol{B}}_1+L_2\bar{\boldsymbol{B}}_2+L_3\bar{\boldsymbol{B}}_3) \\ &= z\sum_{j=1}^{3}L_j\bar{\boldsymbol{T}}\bar{\boldsymbol{B}}_j\end{aligned} \quad (19)$$

where,

$$\bar{\boldsymbol{T}} = \frac{1}{4A^2}\begin{bmatrix} b_1^2 & b_2^2 & b_1b_2 \\ c_1^2 & c_2^2 & c_1c_2 \\ 2b_1c_1 & 2b_2c_2 & b_1c_2+b_2c_1 \end{bmatrix} \quad (20)$$

$$\bar{\boldsymbol{B}}_1 = \boldsymbol{B}(1,0,0,1),\quad \bar{\boldsymbol{B}}_2 = \boldsymbol{B}(0,1,0,1),\quad \bar{\boldsymbol{B}}_3 = \boldsymbol{B}(0,0,1,1) \quad (21)$$

$$\bar{\boldsymbol{B}}_1 = \begin{bmatrix} 6 & 4b_2 & 4c_2 & 0 & 0 & 0 & -6 & 2b_2 & 2c_2 \\ 4 & b_2-b_3 & c_2-c_3 & -2 & -b_3-b_1 & -c_3-c_1 & -2 & b_2+b_1 & c_2+c_1 \\ 4 & 5b_2+3b_3 & 5c_2+3c_3 & 4 & b_3-b_1 & c_3-c_1 & -8 & 3b_2+b_1 & 3c_2+c_1 \end{bmatrix} \quad (22)$$

$$\bar{\boldsymbol{B}}_2 = \begin{bmatrix} -2 & b_2+b_3 & c_2+c_3 & 4 & b_3-b_1 & c_3-c_1 & 2 & -b_2-b_1 & -c_2-c_1 \\ 0 & 0 & 0 & 6 & -4b_1 & -4c_1 & -6 & -2b_1 & -2c_1 \\ 4 & b_2-b_3 & c_2-c_3 & 4 & -3b_3-5b_1 & -3c_3-5c_1 & -8 & -b_2-3b_1 & -3c_2-c_1 \end{bmatrix} \tag{23}$$

$$\bar{\boldsymbol{B}}_3 = \begin{bmatrix} -6 & -2b_2 & -2c_2 & 0 & 0 & 0 & 6 & -4b_2 & -4c_2 \\ 0 & 0 & 0 & -6 & 2b_1 & 2c_1 & 6 & 4b_1 & 4c_1 \\ -4 & -b_2+b_3 & -c_2+c_3 & -4 & -b_3+b_1 & -c_3+c_1 & 8 & 3b_1-3b_2 & 3c_1-3c_2 \end{bmatrix} \tag{24}$$

In this section, we give a very simple formula of geometric matrix with explicit L_i. In this form, it is easy to integral in following derivation. The strains on the surface of the plate at the nodes are given by (18) when L_i=1, z=h/2,

$$\varepsilon_e^0 = \frac{1}{6} h \bar{\boldsymbol{T}} \sum_{i=1}^{3} \bar{\boldsymbol{B}}_i \boldsymbol{U}_e , \tag{25}$$

Nodal strains are obtained by average strain of elements around the nodes.

$$\varepsilon_j = \frac{1}{n_j} \sum_{e \in S_j} \varepsilon_e^0 , \tag{26}$$

where S_j is the set of elements around node j, n_j is the number of elements around node j.

Elastic Matrix and Stresses

If the densities and orientation of beams at node j are denoted as t_{1j}, t_{2j} and α_j, respectively, the elastic matrix at node j is expressed as

$$\boldsymbol{D}_j = \boldsymbol{D}(t_{1j}, t_{2j}, \alpha_j) = E \sum_{b=1}^{2} t_{bj} \sum_{r=1}^{3} s_{br} g_r(\alpha_j) \boldsymbol{A}_r , \tag{27}$$

The elastic matrix at any point in an element can be obtained through standard element interpolation by shape functions,

$$\boldsymbol{D}_e(L_1, L_2) = \sum_{i=1}^{3} L_i \boldsymbol{D}_{j_i} , \tag{28}$$

where j_i is the nodes belong element e. Further the stress at node j is calculated by,

$$\boldsymbol{\sigma}_j = \boldsymbol{D}_j \varepsilon_j . \tag{29}$$

Element Stiffness Matrix of Grillages-Like Continuum at any Point

With the aids of (28) the element stiffness matrix is calculated by,

$$\begin{aligned} \boldsymbol{k}_e &= \int_{V_e} \boldsymbol{B}^{\mathrm{T}} \boldsymbol{D}_e \boldsymbol{B} \, \mathrm{d}V = \\ & A \sum_{r,s,j=1}^{3} \int_{-h/2}^{h/2} z^2 \, \mathrm{d}z \int_0^1 \int_0^{1-L_2} (L_r \bar{\boldsymbol{T}} \bar{\boldsymbol{B}}_r)^{\mathrm{T}} (L_j \boldsymbol{D}_j)(L_s \bar{\boldsymbol{T}} \bar{\boldsymbol{B}}_s) \, \mathrm{d}L_1 \, \mathrm{d}L_2 \\ &= \frac{1}{12} A h^3 \sum_{r,s,j=1}^{3} (\bar{\boldsymbol{T}} \bar{\boldsymbol{B}}_r)^{\mathrm{T}} \boldsymbol{D}_j (\bar{\boldsymbol{T}} \bar{\boldsymbol{B}}_s) \int_0^1 \int_0^{1-L_2} L_r L_j L_s \, \mathrm{d}L_1 \, \mathrm{d}L_2 \\ &= \sum_{r,s,j=1}^{3} I_{rjs} \bar{\boldsymbol{B}}_r^{\mathrm{T}} \bar{\boldsymbol{T}}^{\mathrm{T}} \boldsymbol{D}_j \bar{\boldsymbol{T}} \bar{\boldsymbol{B}}_s \end{aligned} \tag{30}$$

where

$$I_{rjs} = \frac{1}{12} A h^3 \int_0^1 \int_0^{1-L_2} L_r L_j L_s \, \mathrm{d}L_1 \, \mathrm{d}L_2 = \begin{cases} Ah^3/240 & \text{if } r = s = j \\ Ah^3/1440 & \text{if } r \neq s \neq j \\ Ah^3/720 & \text{else} \end{cases} , \tag{31}$$

Material Volume

The material volume is taken as the objective function

$$V = \sum_e \int_{A_e} ht \, \mathrm{d}A \tag{32}$$

where h and t are thickness and density of beam, respectively. For convenience the dimensionless volume is defined as,

$$\bar{V} = \frac{\sigma_p h}{FL^2} V \text{ for example 1, or } \bar{V} = \frac{\sigma_p h}{qR^4} V \text{ for example 2 and 3,} \tag{33}$$

where F and q the magnitude of point load and concentration of uniformly distributed load, respectively. When triangle elements are used, the volume is calculated by,

$$V = \frac{1}{3} h \sum_e A_e \sum_{j \in S_e} \sum_{b=1,2} t_{bj} \tag{34}$$

4. PROCEDURE TO OPTIMIZE DISTRIBUTION OF BEAM

Based upon the above development, the following procedure is proposed to optimize the beam distribution.

(i) Design domain is divided by triangle finite elements and the initial values of design variables are assumed as,

$$t_{bj}^0 = 1 , \ \alpha_j^0 = 0 , \ b=1,2; \ j=1,2,\ldots,J; \tag{35}$$

(ii) Finite elements analysis is performed and the principal stresses and their corresponding directions at nodes are calculated;

(iii) Beams are oriented along the principal stress directions. The beams densities are adjusted on the fully-stressed criterion,

$$\bar{t}_{bj}^{i+1} = t_{bj}^i \sigma_{bj}^i / \sigma_p , \ b=1,2; \ j=1,2,\ldots,J, \tag{36}$$

where the superscript i is the index of iteration, σ_p permissible stress. To prevent the stiffness singularity, too little densities should be avoided by,

$$t_{bj}^{i+1} = \max(t_{th}^{i+1}, \bar{t}_{bj}^{i+1}) , \ b=1,2; \ j=1,2,\ldots,J, \tag{37}$$

where t_{th}^{i+1} being the threshold density defined as,

$$t_{th}^{i+1} = r_1 \times \max_{b,j}(\bar{t}_{bj}^{i+1}) , \tag{38}$$

where the tolerance r_1=10^{-7} is adopted in this work.

(iv) The convergence is checked by comparing the maximum relative densities changes of two successive iterations,

$$\delta^i = \max_{b,j}\left|t_{bj}^{i+1} - t_{bj}^{i}\right| / \max_{b,j}(t_{bj}^{i+1}) < r_2\,, \tag{39}$$

where r_2 is a specified tolerance value and r_2=10^{-2} is used her.

Iteration is terminated if (39) is satisfied, otherwise return to (ii).

After the optimal material fields are determined the optimal discrete beams system can be constructed.

5. NUMERICAL EXAMPLES

The mechanics models of three examples are shown in Fig. (**1**). The main parameters are given in Table **1**. 3-nodes triangle plate elements are used. The $L \times L$ square domain is shown in example 1. All four boundaries are simple supported. A point load acted at center. Due to its symmetry a quarter of the structure is analyzed and illustrated in following. In example 2, a quarter of round plate is loaded by uniformed distributed force. Its two straight boundaries are simple supported and the quarter of circle is free. Its two right angle boundaries are simple supported and the circumference boundary is free. In example 3, a half round plate is loaded by uniformed distributed force. Its half circumference boundaries are simple supported and its straight boundary is free. The finite element division patterns are showed in Fig (**2**). To compare the method with different elements patterns, two kinds of element patterns are adopted in example 1. The topology optimal structure is independence of magnitude of the magnitudes

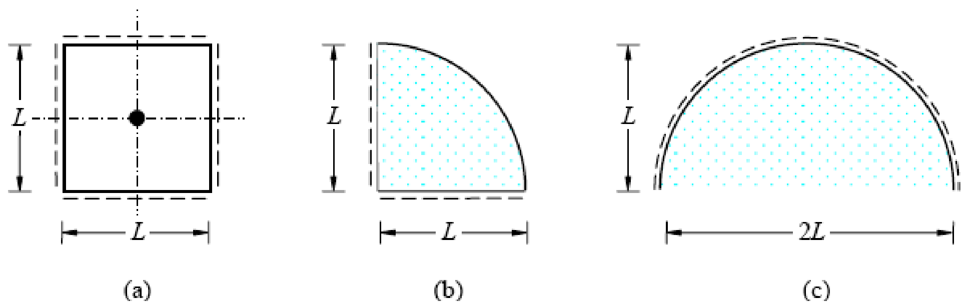


Fig. (1). Mechanics models. (**a**) Example 1, (**b**) Example 2, (**c**) Example 3.

Table 1. Parameter of Examples

	Example 1				
Elements	**Pattern a**	**Pattern b**	**Rectangular**	**Example 2**	**Example 3**
Number of elements	512	1024	256	104	104
Number of nodes	289	545	289	67	67
Number of iteration	11	15	46	17	23
Stress error%	10.3	4.56	2.56	3.78	3.54
Dimensionless Volume	0.74977	0.75000	0.75279	0.74516	1.17793
Exact Volume	0.75			-	-

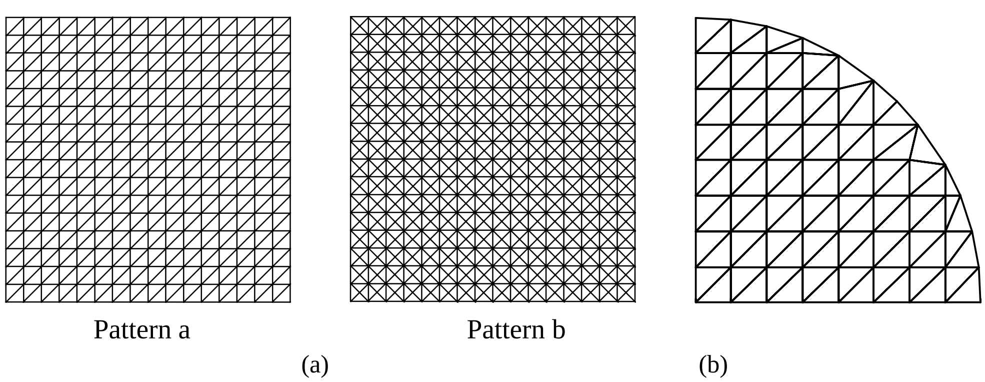
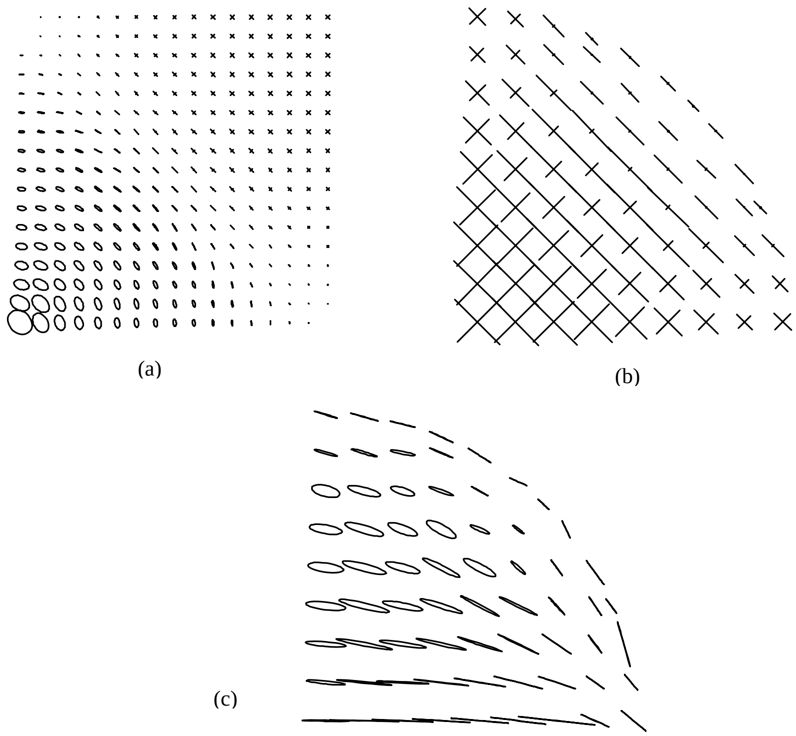


Fig. (2). Finite Elements. (**a**) Two patterns of elements in example 1, (**b**) Example 2 and Example 3.

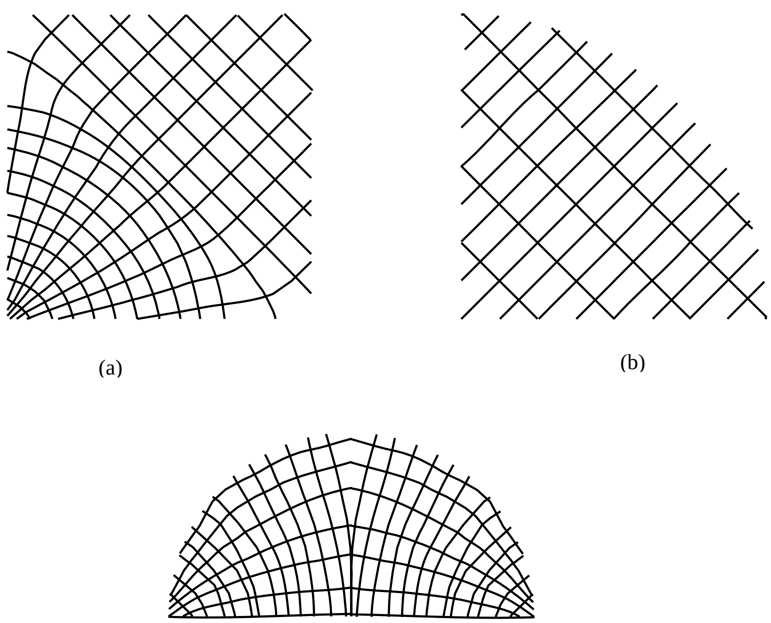
of loads and sizes; therefore their concrete values are not given here.

The optimum moment and material distributions are showed in Fig (**3**). The ellipses represent the moments with same signs in all directions; the crosses denote the moments with different signs in two orthotropic directions. The lengths and directions of two principal axes of ellipses (or cross lines) stand for the magnitudes and directions of principal moments. Based on the moment distribution field, a discrete beam system is constructed as showed in Fig. (**4**). It should be noted that equivalent discrete structures are not unique. The whole procedures, including all figures, are achieved by program automatically. The iteration histories of volume of structures are shown in Fig. (**5**). To compare the



(a) (b) (c)

Fig. (3). Optimal beams distributed fields. (**a**) Example 1 ,a quarter of the structure is analyzed for symmetry; (**b**) Example 2, (**c**) Example 3 the half structure is analyzed for symmetry.



(a) (b) (c)

Fig. (4). Suggested optimal beams systems. (**a**) Example 1, (**b**) Example 2, (**c**) Example 3.

optimization processes with different elements, the result of rectangular element is shown here. It is shown that the material volume optimized by triangular elements is less than that by rectangular elements [15]. In iteration process, stresses tend to be fully stressed. Their relative errors to fully stress are shown in Fig. (**6**). The relative errors of triangular elements are less than that of rectangular elements. Of course, the freedom of triangular elements in pattern 2 is great than that of rectangular elements. Additionally, for the stiffness of triangular elements being higher than that of rectangular elements, the material volume of triangular elements is less, as shown in Table **1**. The exact analytical solutions of example 1 are showed in Fig. (**7a**), which give the optimized distribution of members. For example, the simple supported beam shown in Fig. (**7b**) is one of optimized structures.

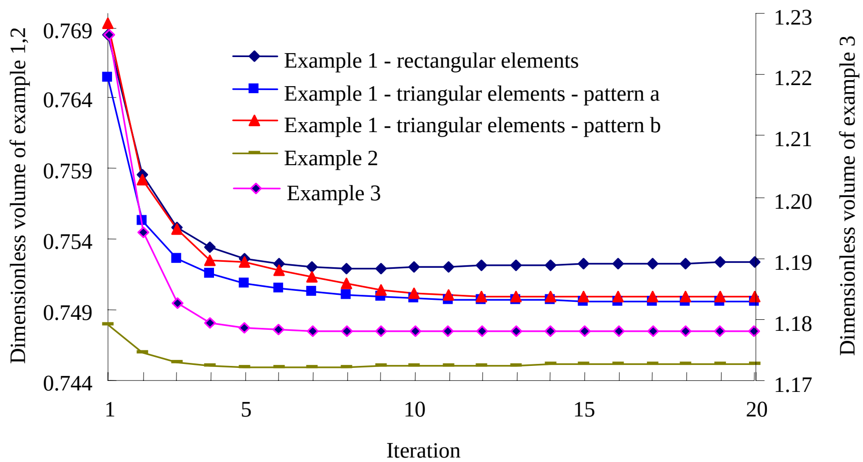


Fig. (5). Convergence process of material volume.

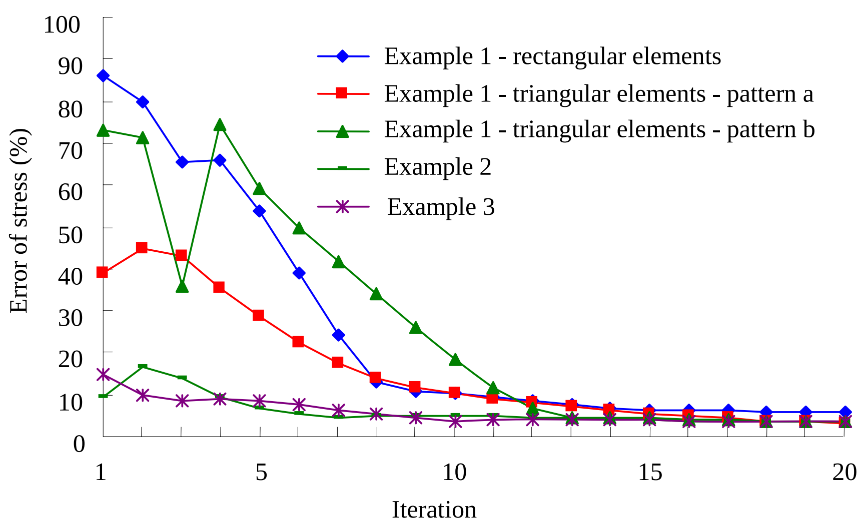


Fig. (6). Convergence process of stress error.

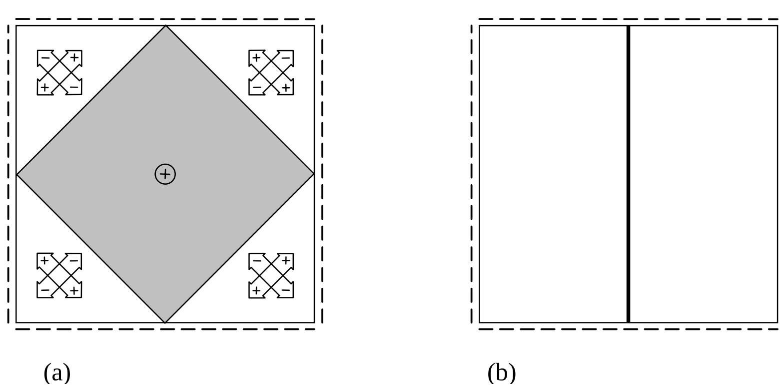


Fig. (7). Exact solution of example 1 (**a**) optimized distribution of member; (**b**) One of optimization structure.

Exact solution of example 1 was given by Rozvany [2] and Morley [4], and is shown in Fig. (**6**). In the central region marked by $\oplus$ beams along any orientation are optimal. In other region optimal beams are lay along the arrowhead. The sign "+" and "-" means the moments of beams.

CONCLUSIONS

A numerical method based on triangular elements for the topology optimization of least-weight grillages was proposed. In contrast to most numerical method which directly leads to discrete structure, a material distribution field is firstly obtained which subsequently serves the basis to construct the discrete structures. A very simple geometry matrix of triangle element is given. The optimized results based on triangular elements are better than that on rectangular elements. The stiffness matrix of truss-like continuum in triangle element is derived. This method is robust and can handle arbitrary loading and boundary conditions and shape.

CONFLICT OF INTEREST

Declared none.

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